CS 526

Advanced Compiler Construction

http://misailo.cs.Illinois.edu/courses/cs526

DATAFLOW ANALYSIS

The slides adapted from Martin Rinard and Vikram Adve

Comparison

Reaching Definitions

```
for all nodes n in N
    OUT[n] = emptyset;
IN[Entry] = emptyset;
OUT[Entry] = GEN[Entry];
Changed = N - { Entry };

while (Changed != emptyset)
    choose a node n in Changed;
    Changed = Changed - { n };

    IN[n] = emptyset;
    for all nodes p in predecessors(n)
        IN[n] = IN[n] U OUT[p];

    OUT[n] = GEN[n] U (IN[n] - KILL[n]);

    if (OUT[n] changed)
        for all nodes s in successors(n)
            Changed = Changed U { s };
```

Available Expressions

```
for all nodes n in N
    OUT[n] = E;
IN[Entry] = emptyset;
OUT[Entry] = GEN[Entry];
Changed = N - { Entry };

while (Changed != emptyset)
    choose a node n in Changed;
    Changed = Changed - { n };

    IN[n] = E;
    for all nodes p in predecessors(n)
        IN[n] = IN[n] ∩ OUT[p];

    OUT[n] = GEN[n] U (IN[n] - KILL[n]);

    if (OUT[n] changed)
            for all nodes s in successors(n)
            Changed = Changed U { s };
```

Liveness

```
for all nodes n in N - { Exit }
    IN[n] = emptyset;
OUT[Exit] = emptyset;
IN[Exit] = use[Exit];
Changed = N - { Exit };

while (Changed != emptyset)
    choose a node n in Changed;
    Changed = Changed - { n };

    OUT[n] = emptyset;
    for all nodes s in successors(n)
        OUT[n] = OUT[n] U IN[p];

    IN[n] = use[n] U (out[n] - def[n]);

    if (IN[n] changed)
        for all nodes p in predecessors(n)
            Changed = Changed U { p };
```

Comparison

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Liveness

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while (Changed != emptyset)
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    Changed = Changed - { n };

    OUT[n] = emptyset;
    for all nodes s in successors(n)
        OUT[n] = OUT[n] U IN[p];

    IN[n] = use[n] U (out[n] - def[n]);

    if (IN[n] changed)
        for all nodes p in predecessors(n)
            Changed = Changed U { p };
```

Basic Idea

Information about program represented using values from algebraic structure called **lattice**

Analysis produces lattice value for each program point

Two flavors of analysis

- Forward dataflow analysis [e.g., Reachability]
- Backward dataflow analysis [e.g. Live Variables]

Program Representation (Reminder)

Control Flow Graph

- Nodes N – statements of program
- Edges E – flow of control
  - pred(n) = set of all predecessors of n
  - succ(n) = set of all successors of n
- Start node n_0
- Set of final nodes N_{final}

Program Points

- One program point before each node
- One program point after each node
- Join point – point with multiple predecessors
- Split point – point with multiple successors

Transfer function

- Propagates dataflow facts through the basic blocks

Partial Orders

Set P

Partial order relation $\leq$ such that $\forall x,y,z \in P$

- $x \leq x$ (reflexive)
- $x \leq y$ and $y \leq x$ implies $x = y$ (asymmetric)
- $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitive)

Can use partial order to define

- Upper and lower bounds
- Least upper bound
- Greatest lower bound

Upper Bounds

If $S \subseteq P$ then

- $x \in P$ is an upper bound of S if $\forall y \in S.\ y \leq x$
- $x \in P$ is the least upper bound of S if
  - x is an upper bound of S, and
  - $x \leq y$ for all upper bounds y of S
- $\vee$ **- join**, least upper bound, **lub**, supremum, **sup**
  - $\vee$ S is the least upper bound of S
  - $x \vee y$ is the least upper bound of {x,y}

Lower Bounds

If $S \subseteq P$ then

- $x \in P$ is a lower bound of S if $\forall y \in S.\ x \leq y$
- $x \in P$ is the greatest lower bound of S if
  - x is a lower bound of S, and
  - $y \leq x$ for all lower bounds y of S
- $\wedge$ **- meet**, greatest lower bound, **glb**, infimum, **inf**
  - $\wedge$ S is the greatest lower bound of S
  - $x \wedge y$ is the greatest lower bound of {x,y}

Covering

$x < y$ if $x \leq y$ and $x \neq y$

x is covered by y (y covers x) if

- $x < y$, and
- $x \leq z < y$ implies $x = z$

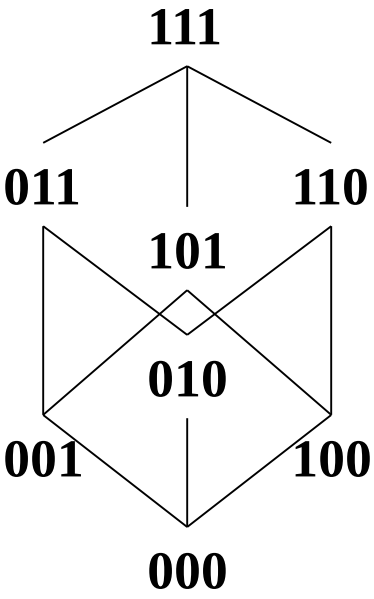
Conceptually, y covers x if there are no elements between x and y

Example

P = { 000, 001, 010, 011, 100, 101, 110, 111}

(standard boolean lattice, also called **hypercube**)

$x \leq y$ **if** (x bitwise and y) = x



Hasse Diagram

- If y covers x
  - Line from y to x
  - y above x in diagram

Lattices

If $x \wedge y$ and $x \vee y$ exist for all $x,y \in P$,
then P is a lattice.

If $\wedge S$ and $\vee S$ exist for all $S \subseteq P$,
then P is a complete lattice.

All finite lattices are complete

Greatest element of P (if it exists) is top ($\top$)

Least element of P (if it exists) is bottom ($\bot$)

Incomplete Lattice

Example of a lattice that is not complete:

Integers I

- For any $x, y \in I$, $x \vee y = \max(x,y)$, $x \wedge y = \min(x,y)$
- But $\vee I$ and $\wedge I$ do not exist
- $I \cup \{+\infty, -\infty\}$ is a complete lattice

Connection Between $\leq$, $\wedge$, and $\vee$

The following 3 properties are equivalent:

- $x \leq y$
- $x \vee y = y$
- $x \wedge y = x$

Will prove:

- $x \leq y$ implies $x \vee y = y$ and $x \wedge y = x$
- $x \vee y = y$ implies $x \leq y$
- $x \wedge y = x$ implies $x \leq y$

Then by transitivity, can obtain

- $x \vee y = y$ implies $x \wedge y = x$
- $x \wedge y = x$ implies $x \vee y = y$

Connecting Lemma Proofs

Prove: $x \leq y$ implies $x \vee y = y$

- $x \leq y$ implies y is an upper bound of {x,y}.
- Any upper bound z of {x,y} must satisfy $y \leq z$.
- So y is least upper bound of {x,y} and $x \vee y = y$

Prove: $x \leq y$ implies $x \wedge y = x$

- $x \leq y$ implies x is a lower bound of {x,y}.
- Any lower bound z of {x,y} must satisfy $z \leq x$.
- So x is greatest lower bound of {x,y} and $x \wedge y = x$

Connecting Lemma Proofs

Prove: $x \vee y = y$ implies $x \leq y$

- y is an upper bound of {x,y} implies $x \leq y$

Prove: $x \wedge y = x$ implies $x \leq y$

- x is a lower bound of {x,y} implies $x \leq y$

Lattices as Algebraic Structures

We have defined $\vee$ and $\wedge$ in terms of $\leq$

We will now define $\leq$ in terms of $\vee$ and $\wedge$

- Start with $\vee$ and $\wedge$ as arbitrary algebraic operations that satisfy associative, commutative, idempotence, and absorption laws
- Will define $\leq$ using $\vee$ and $\wedge$
- Will show that $\leq$ is a partial order

Intuitive concept of $\vee$ and $\wedge$ as information combination operators (or, and)

Algebraic Properties of Lattices

Assume arbitrary operations $\vee$ and $\wedge$ such that

- $(x \vee y) \vee z = x \vee (y \vee z)$ (associativity of $\vee$)
- $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ (associativity of $\wedge$)
- $x \vee y = y \vee x$ (commutativity of $\vee$)
- $x \wedge y = y \wedge x$ (commutativity of $\wedge$)
- $x \vee x = x$ (idempotence of $\vee$)
- $x \wedge x = x$ (idempotence of $\wedge$)
- $x \vee (x \wedge y) = x$ (absorption of $\vee$ over $\wedge$)
- $x \wedge (x \vee y) = x$ (absorption of $\wedge$ over $\vee$)

Connection Between $\wedge$ and $\vee$

$x \vee y = y$ if and only if $x \wedge y = x$

Proof of $x \vee y = y$ implies $x = x \wedge y$

$x = x \wedge (x \vee y)$ (by absorption)

$= x \wedge y$ (by assumption)

Proof of $x \wedge y = x$ implies $y = x \vee y$

$y = y \vee (y \wedge x)$ (by absorption)

$= y \vee (x \wedge y)$ (by commutativity)

$= y \vee x$ (by assumption)

$= x \vee y$ (by commutativity)

Properties of $\leq$

Define $x \leq y$ if $x \vee y = y$

Proof of transitive property. Must show that

$x \vee y = y$ and $y \vee z = z$ implies $x \vee z = z$

$x \vee z = x \vee (y \vee z)$ (by assumption)

$= (x \vee y) \vee z$ (by associativity)

$= y \vee z$ (by assumption)

$= z$ (by assumption)

Properties of $\leq$

Proof of asymmetry property. Must show that

$x \vee y = y$ and $y \vee x = x$ implies $x = y$

$x = y \vee x$ (by assumption)

$= x \vee y$ (by commutativity)

$= y$ (by assumption)

Proof of reflexivity property. Must show that

$x \vee x = x$

$x \vee x = x$ (by idempotence)

Properties of $\leq$

Induced operation $\leq$ agrees with original definitions of $\vee$ and $\wedge$, i.e.,

- $x \vee y = \sup \{x, y\}$
- $x \wedge y = \inf \{x, y\}$

Proof of $x \vee y = \sup \{x, y\}$

Consider any upper bound u for x and y.

Given $x \vee u = u$ and $y \vee u = u$, must show

$x \vee y \leq u$, i.e., $(x \vee y) \vee u = u$

$u = x \vee u$ (by assumption)

$= x \vee (y \vee u)$ (by assumption)

$= (x \vee y) \vee u$ (by associativity)

Proof of $x \wedge y = \inf \{x, y\}$

- Consider any lower bound l for x and y.
- Given $x \wedge l = l$ and $y \wedge l = l$, must show $l \leq x \wedge y$, i.e., $(x \wedge y) \wedge l = l$

$$
\begin{aligned}
l &= x \wedge l && \text{(by assumption)} \\
&= x \wedge (y \wedge l) && \text{(by assumption)} \\
&= (x \wedge y) \wedge l && \text{(by associativity)}
\end{aligned}
$$

Chains

A set S is a chain if $\forall x,y \in S.\ y \leq x$ or $x \leq y$

P has no infinite chains if every chain in P is finite

P satisfies the ascending chain condition if for all sequences $x_1 \leq x_2 \leq \ldots$ there exists n such that $x_n = x_{n+1} = \ldots$

Application to Dataflow Analysis

Dataflow information will be lattice values

- **Transfer functions** operate on lattice values
- Solution algorithm will generate **increasing sequence of values** at each program point
- Ascending chain condition will ensure **termination**

We will use $\vee$ to combine values at control-flow join points

Transfer Functions

Transfer function $f: P \rightarrow P$ for each node in control flow graph

f models effect of the node on the program information

Transfer Functions

Each dataflow analysis problem has a **set F of transfer functions** f: $P \to P$, i.e.,

- **Identity function** belongs to the set, $i \in F$
- F must be **closed under composition**:
  $\forall f,g \in F$. the function $h = \lambda x.f(g(x)) \in F$
- Each $f \in F$ must be **monotone**:
  $x \leq y$ implies $f(x) \leq f(y)$
- Sometimes all $f \in F$ are **distributive**:
  $f(x \vee y) = f(x) \vee f(y)$
- Note that Distributivity implies monotonicity

Distributivity Implies Monotonicity

Proof.

Assume distributivity: $f(x \vee y) = f(x) \vee f(y)$

Must show: $x \vee y = y$ implies $f(x) \vee f(y) = f(y)$

$$f(y) = f(x \vee y) \quad \text{(by assumption)}$$
$$= f(x) \vee f(y) \quad \text{(by distributivity)}$$

Putting Pieces Together

Forward Dataflow Analysis Framework

Simulates execution of program forward with flow of control

Forward Dataflow Analysis

Simulates execution of program forward with flow of control

For each node n, have

- in_n – value at program point before n
- out_n – value at program point after n
- f_n – transfer function for n (given in_n, computes out_n)

Require that solution satisfy

- $\forall n.\ out_n = f_n(in_n)$
- $\forall n \neq n_0.\ in_n = \vee \{\ out_m\ .\ m \text{ in } pred(n)\ \}$
- $in_{n0} = I$
- Where I summarizes information at start of program

Dataflow Equations

Compiler processes program to obtain a set of dataflow equations

$$out_n := f_n(in_n)$$

$$in_n := \vee \{ out_m \,.\, m \text{ in } pred(n) \}$$

Conceptually separates analysis problem from program

Worklist Algorithm for Solving Forward Dataflow Equations

for each n do $out_n := f_n(\bot)$

$in_{n0} := I$; $out_{n0} := f_{n0}(I)$

worklist := N - { n_0 }

while worklist $\neq \emptyset$ do

    remove a node n from worklist

    $in_n := \vee$ { out_m . m in pred(n) }

    $out_n := f_n(in_n)$

    if out_n changed then

        worklist := worklist $\cup$ succ(n)

Correctness Argument

Why does result satisfy dataflow equations?

Whenever process a node n, algorithm sets $out_n := f_n(in_n)$
Algorithm ensures that $out_n = f_n(in_n)$

Whenever out_m changes, put succ(m) on worklist.
Consider any node $n \in succ(m)$. It will eventually come off worklist and algorithm will set

$$in_n := \vee \{ out_m \,.\, m \text{ in } pred(n) \}$$

to ensure that $in_n = \vee \{ out_m \,.\, m \text{ in } pred(n) \}$

So final solution will satisfy dataflow equations

Termination Argument

Why does algorithm terminate?

Sequence of values taken on by in_n or out_n is a chain. If values stop increasing, worklist empties and algorithm terminates.

If lattice has ascending chain property, algorithm terminates

- **Algorithm terminates for finite lattices**
- For lattices without ascending chain property, use widening operator

Widening Operators

Detect lattice values that may be part of infinitely ascending chain

Artificially raise value to least upper bound of chain

Example:

- Lattice is set of all subsets of integers
- Could be used to collect possible values taken on by variable during execution of program
- Widening operator might raise all sets of size n or greater to TOP (likely to be useful for loops)

Reaching Definitions Algorithm

Reminder

```
for all nodes n in N
      OUT[n] = emptyset; // OUT[n] = GEN[n];
IN[Entry] = emptyset;
OUT[Entry] = GEN[Entry];
Changed = N - { Entry }; // N = all nodes in graph

while (Changed != emptyset)
      choose a node n in Changed;
      Changed = Changed - { n };

      IN[n] = emptyset;
      for all nodes p in predecessors(n)
            IN[n] = IN[n] U OUT[p];

      OUT[n] = GEN[n] U (IN[n] - KILL[n]);

      if (OUT[n] changed)
            for all nodes s in successors(n)
                  Changed = Changed U { s };
```

Reaching Definitions

P = powerset of set of all definitions in program (all subsets of set of definitions in program)

$\vee = \cup$ (order is $\subseteq$)

$\perp = \varnothing$

$I = in_{n0} = \perp$

F = all functions f of the form $f(x) = a \cup (x-b)$

- b is set of definitions that node kills
- a is set of definitions that node generates

General pattern for many transfer functions

- $f(x) = GEN \cup (x-KILL)$

Does Reaching Definitions Framework Satisfy Properties?

$\subseteq$ satisfies conditions for $\leq$

- $x \subseteq y$ and $y \subseteq z$ implies $x \subseteq z$ (transitivity)
- $x \subseteq y$ and $y \subseteq x$ implies $y = x$ (asymmetry)
- $x \subseteq x$ (idempotence)

F satisfies transfer function conditions

- $\lambda x.\varnothing \cup (x - \varnothing) = \lambda x.x \in F$ (identity)
- Will show $f(x \cup y) = f(x) \cup f(y)$ (distributivity)

$$f(x) \cup f(y) = (a \cup (x - b)) \cup (a \cup (y - b))$$
$$= a \cup (x - b) \cup (y - b) = a \cup ((x \cup y) - b)$$
$$= f(x \cup y)$$

Does Reaching Definitions Framework Satisfy Properties?

What about composition?

- Given $f_1(x) = a_1 \cup (x\text{-}b_1)$ and $f_2(x) = a_2 \cup (x\text{-}b_2)$
- Must show $f_1(f_2(x))$ can be expressed as $a \cup (x - b)$

$$f_1(f_2(x)) = a_1 \cup ((a_2 \cup (x\text{-}b_2)) - b_1)$$
$$= a_1 \cup ((a_2 - b_1) \cup ((x\text{-}b_2) - b_1))$$
$$= (a_1 \cup (a_2 - b_1)) \cup ((x\text{-}b_2) - b_1))$$
$$= (a_1 \cup (a_2 - b_1)) \cup (x\text{-}(b_2 \cup b_1))$$

- Let $a = (a_1 \cup (a_2 - b_1))$ and $b = b_2 \cup b_1$
- Then $f_1(f_2(x)) = a \cup (x - b)$

General Result

All GEN/KILL transfer function frameworks satisfy the three properties:

- Identity
- Distributivity
- Composition

Available Expressions

P = powerset of set of all expressions in program (all subsets of set of expressions)

$\vee = \cap$ (order is $\supseteq$)

$\perp = P$

$I = in_{n0} = \varnothing$

F = all functions f of the form $f(x) = a \cup (x\text{-}b)$

- b is set of expressions that node kills
- a is set of expressions that node generates

Another GEN/KILL analysis

Concept of Conservatism

Reaching definitions use $\cup$ as join

- Optimizations must take into account all definitions that reach along ANY path

Available expressions use $\cap$ as join

- Optimization requires expression to reach along ALL paths

Optimizations must **conservatively take all possible executions into account.**

Backward Dataflow Analysis

- Simulates execution of program backward against the flow of control
- For each node n, have
  - in_n – value at program point before n
  - out_n – value at program point after n
  - f_n – transfer function for n (given out_n, computes in_n)
- Require that solution satisfies
  - $\forall n.\ in_n = f_n(out_n)$
  - $\forall n \notin N_{final}.\ out_n = \vee \{\ in_m\ .\ m \text{ in succ}(n)\ \}$
  - $\forall n \in N_{final} = out_n = O$
  - Where O summarizes information at end of program

Worklist Algorithm for Solving Backward Dataflow Equations

for each n do $in_n := f_n(\bot)$

for each $n \in N_{final}$ do $out_n := O$; $in_n := f_n(O)$

worklist := $N - N_{final}$

while worklist $\neq \emptyset$ do

    remove a node n from worklist

    $out_n := \vee \{ in_m . m \text{ in succ(n)} \}$

    $in_n := f_n(out_n)$

    if in_n changed then

        worklist := worklist $\cup$ pred(n)

Live Variables

P = powerset of set of all variables in program (all subsets of set of variables in program)

$\vee = \cup$ (order is $\subseteq$)

$\bot = \varnothing$

$\mathbf{O} = \varnothing$

F = all functions f of the form $f(x) = a \cup (x-b)$

- b is set of variables that node kills
- a is set of variables that node reads

Meaning of Dataflow Results

Concept of program state s for control-flow graphs

- Program point n where execution located (n is node that will execute next)
- Values of variables in program

Each execution generates a trajectory of states:

- $s_0;s_1;\ldots;s_k$, where each $s_i \in ST$
- s_{i+1} generated from s_i by executing basic block to
  - Update variable values
  - Obtain new program point n

Relating States to Analysis Result

- Meaning of analysis results is given by an abstraction function AF:ST$\rightarrow$P
- Correctness condition: require that for all states s

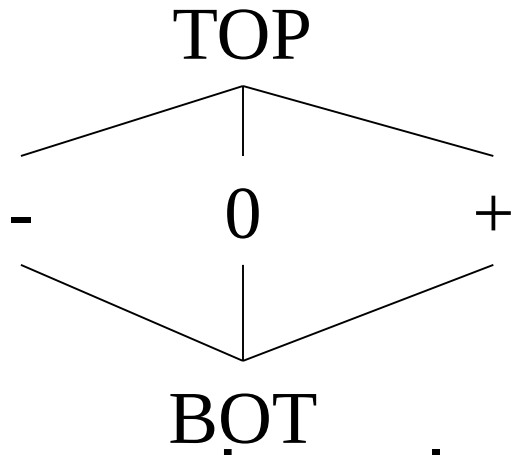
$$AF(s) \leq in_n$$

where n is the next statement to execute in state s

Sign Analysis Example

Sign analysis - compute sign of each variable v

Base Lattice: P = flat lattice on {-,0,+}



Actual lattice records a value for each variable

- Example element: [a→+, b→0, c→-]

Interpretation of Lattice Values

If value of v in lattice is:

- BOT: no information about sign of v
- -: variable v is negative
- 0: variable v is 0
- +: variable v is positive
- TOP: v may be positive or negative

What is abstraction function AF?

- AF($[x_1,\ldots,x_n]$) = $[\text{sign}(x_1), \ldots, \text{sign}(x_n)]$
- Where sign(x) = 0 if x = 0, + if x > 0, - if x < 0

Operation ⊗ on Lattice

⊗	BOT	-	0	+	TOP
BOT	BOT	BOT	0	BOT	BOT
-	BOT	+	0	-	TOP
0	0	0	0	0	0
+	BOT	-	0	+	TOP
TOP	BOT	TOP	0	TOP	TOP

Transfer Functions

If n of the form v = c

- $f_n(x) = x[v \rightarrow +]$ if c is positive
- $f_n(x) = x[v \rightarrow 0]$ if c is 0
- $f_n(x) = x[v \rightarrow -]$ if c is negative

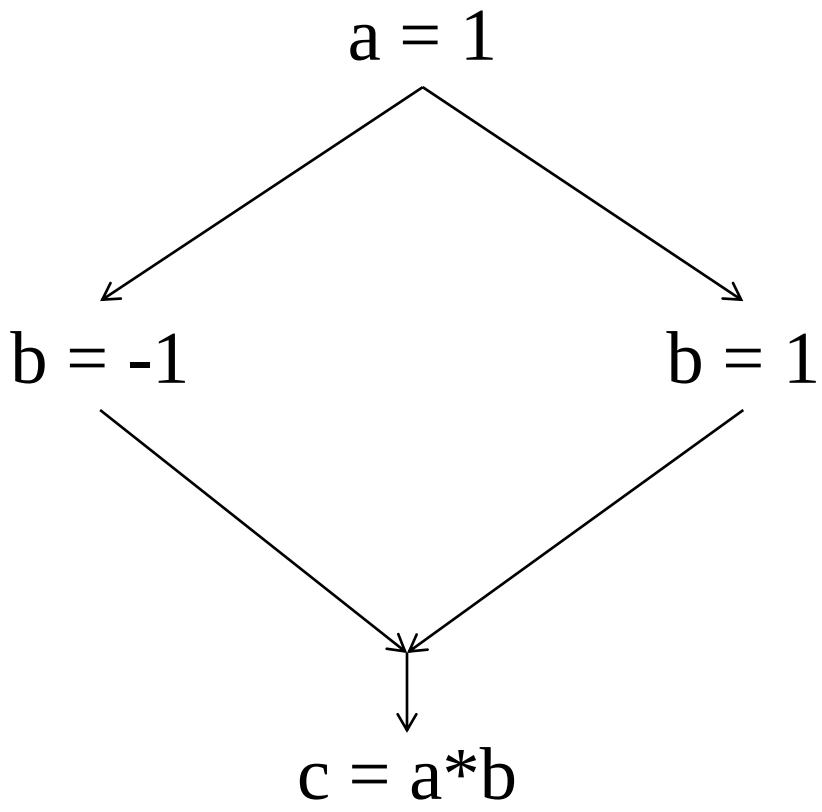
If n of the form $v_1 = v_2 * v_3$

- $f_n(x) = x[v_1 \rightarrow x[v_2] \otimes x[v_3]]$

I = TOP

(uninitialized variables may have any sign)

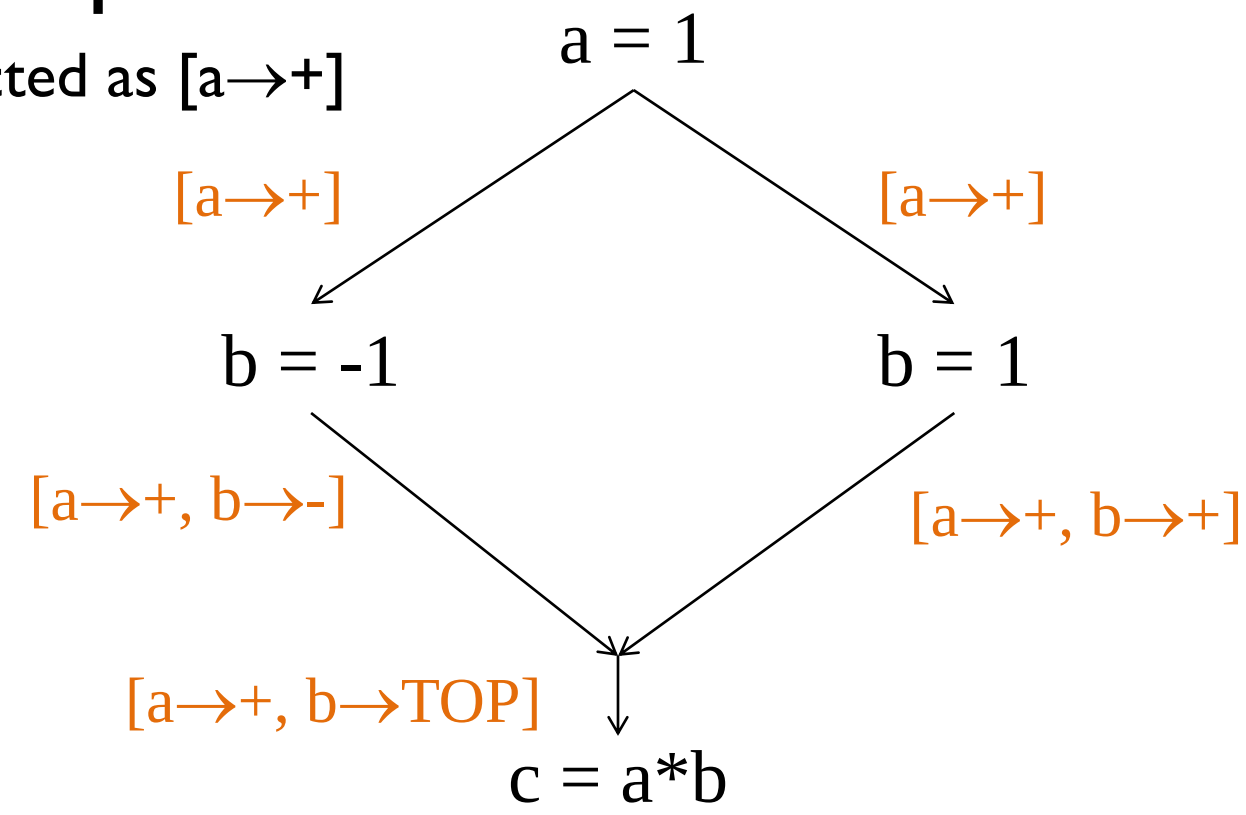
Sign Analysis Example



Imprecision In Example

Abstraction Imprecision:

[a→1] abstracted as [a→+]



Control Flow Imprecision:

[b→TOP] summarizes results of all executions.

(In any concrete execution state s, AF(s)[b]≠TOP)

General Sources of Imprecision

Abstraction Imprecision

- Concrete values (integers) abstracted as lattice values (-,0, and +)
- Lattice values less precise than execution values
- Abstraction function throws away information

Control Flow Imprecision

- One lattice value for all possible control flow paths
- Analysis result has a single lattice value to summarize results of multiple concrete executions
- Join operation $\vee$ moves up in lattice to combine values from different execution paths
- Typically if $x \leq y$, then x is more precise than y

Why To Allow Imprecision?

Make analysis tractable

Unbounded sets of values in execution

- Typically abstracted by finite set of lattice values

Execution may visit unbounded set of states

- Abstracted by computing joins of different paths