Probabilistic & Approximate Computing

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UIUC

Operations on Random Variables

X ~ $\mathcal{N}(0,1)$

Y ~ $\mathcal{N}(0,1)$

X and Y are i.i.d.

Z = X + Y is also a Gaussian – $\mathcal{N}(0,2)$

Why?

Convolution!

$$\int_{-\infty}^{+\infty} f_X(x) \cdot f_Y(z-x)\,dx$$

But How About This?

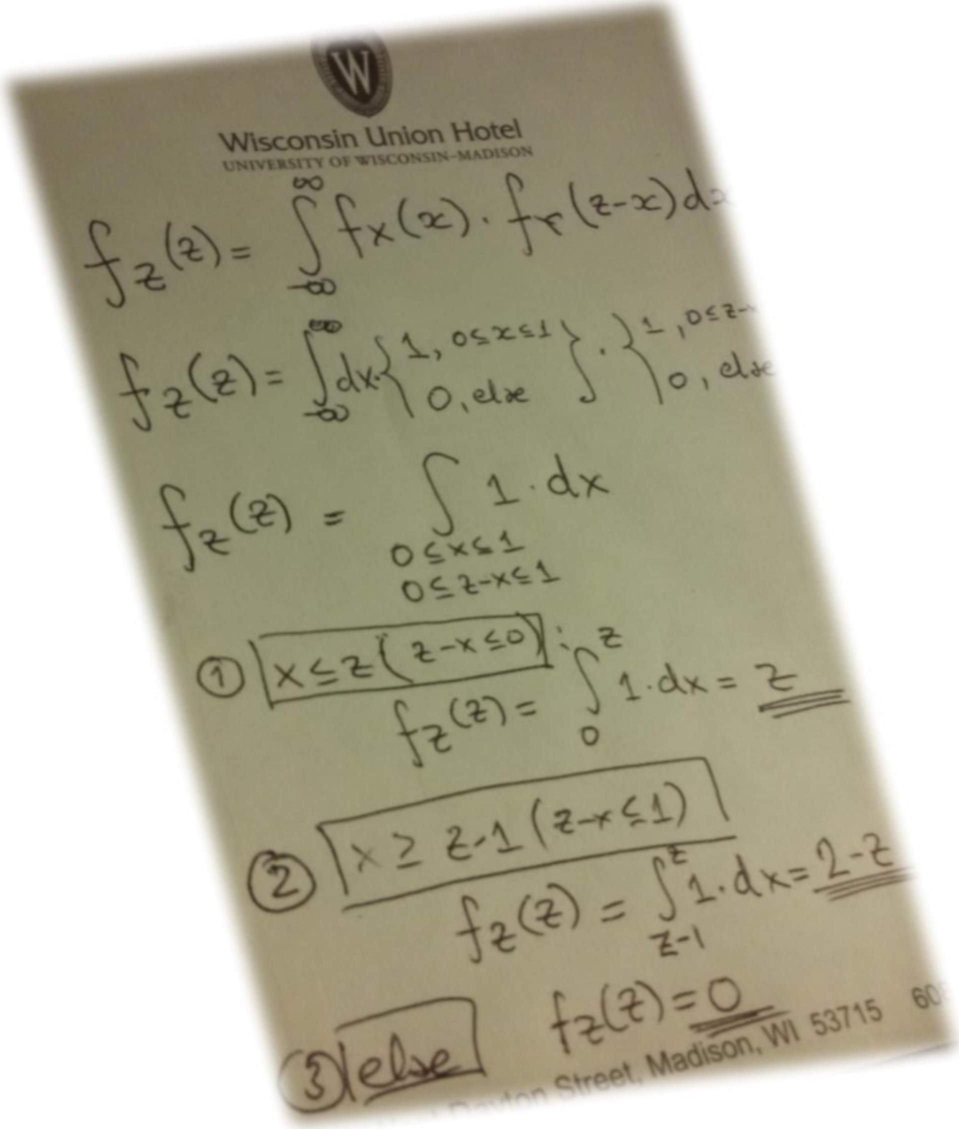
$X \sim \mathcal{U}(0,1)$

$Y \sim \mathcal{U}(0,1)$

X and Y are **i.i.d.**

$Z = X+Y$

What is PDF of Z?



But How About This?

$X \sim \mathcal{U}(0,1)$

$Y \sim \mathcal{U}(0,1)$

X and Y are **i.i.d.**

$Z = X+Y$

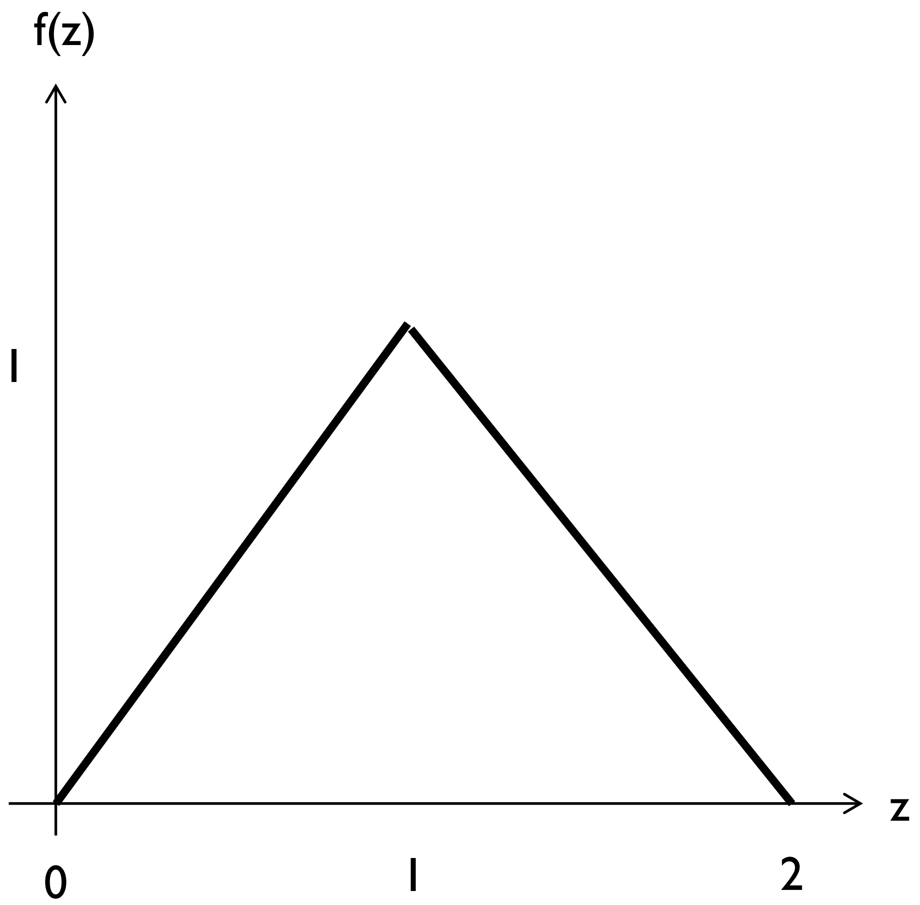
What is PDF of Z?

```
X := Uniform(0,1)
Y := Uniform(0,1)

Z := X + Y

return Z
```

www.psisolver.org

$ psi sum_uniform.prb

But How About This?

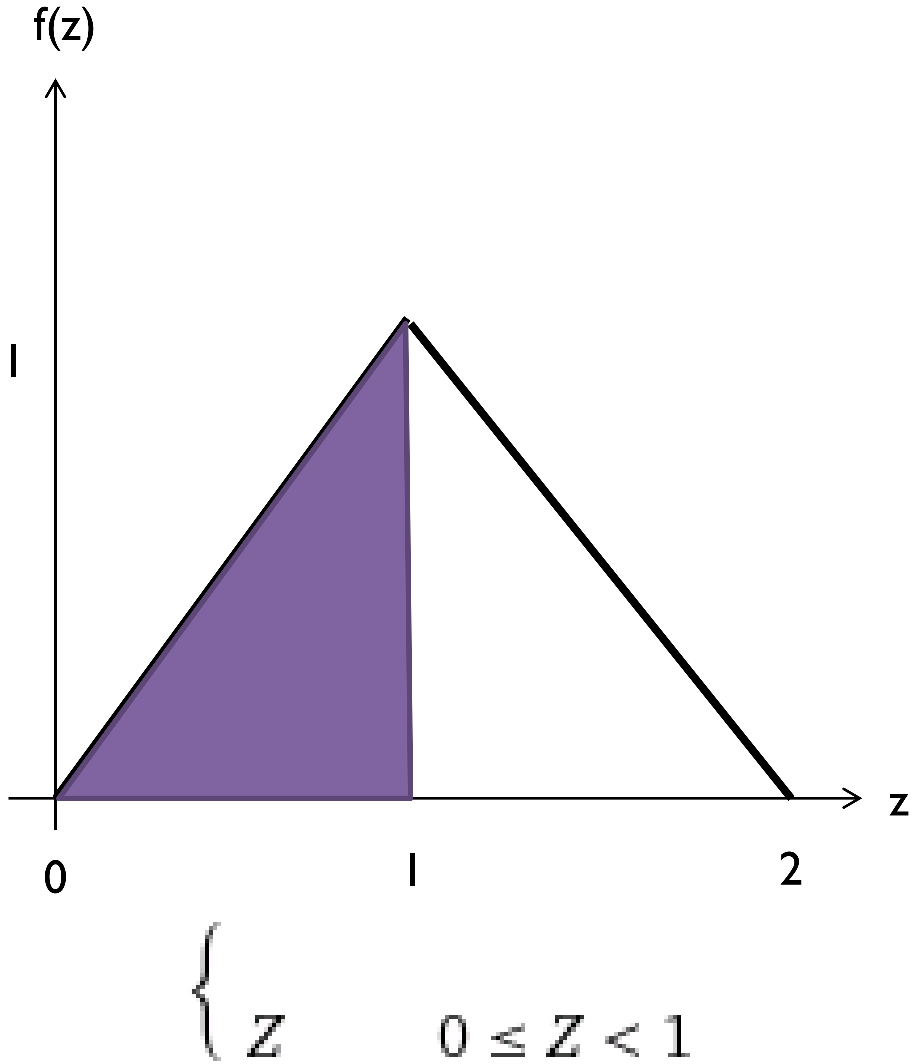


$$\begin{cases} 2 - Z & 1 \le Z \le 2 \\ Z & 0 \le Z < 1 \end{cases}$$

```
X := Uniform(0,1)
Y := Uniform(0,1)

Z := X + Y

return Z
```

$ psi sum_uniform.prb

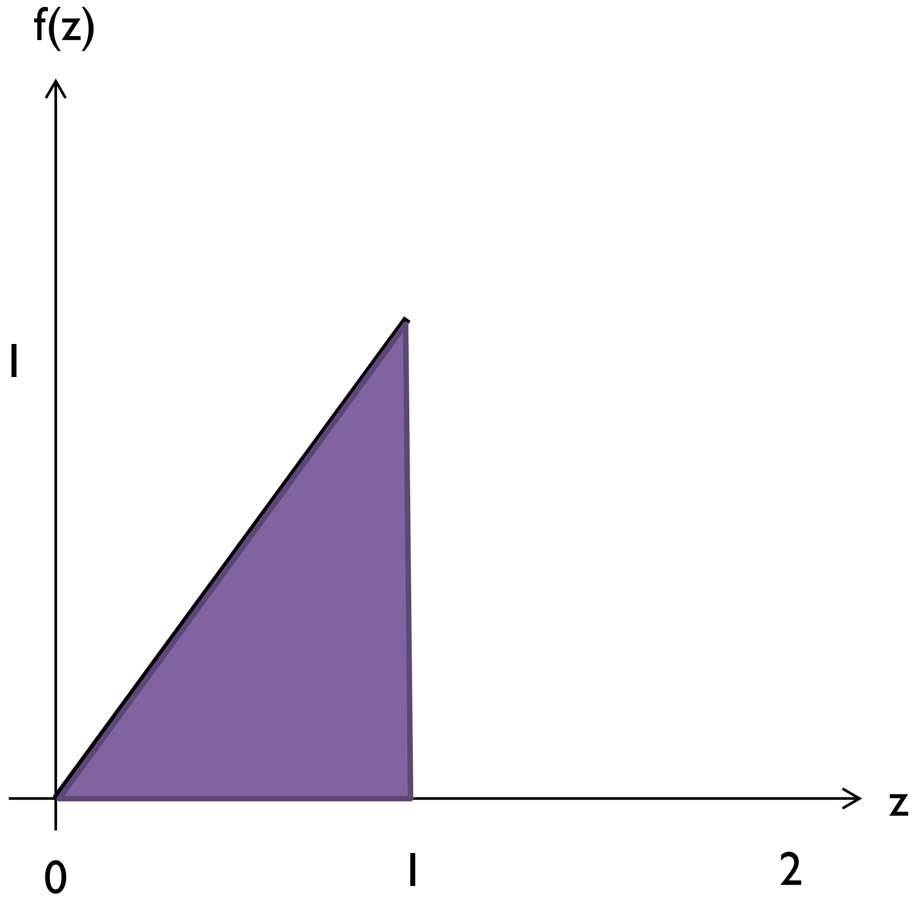
But How About This?




```
X := Uniform(0,1)
Y := Uniform(0,1)

Z := X + Y
observe Z < 1
return Z
```

$ psi sum_uniform.prb

But How About This?



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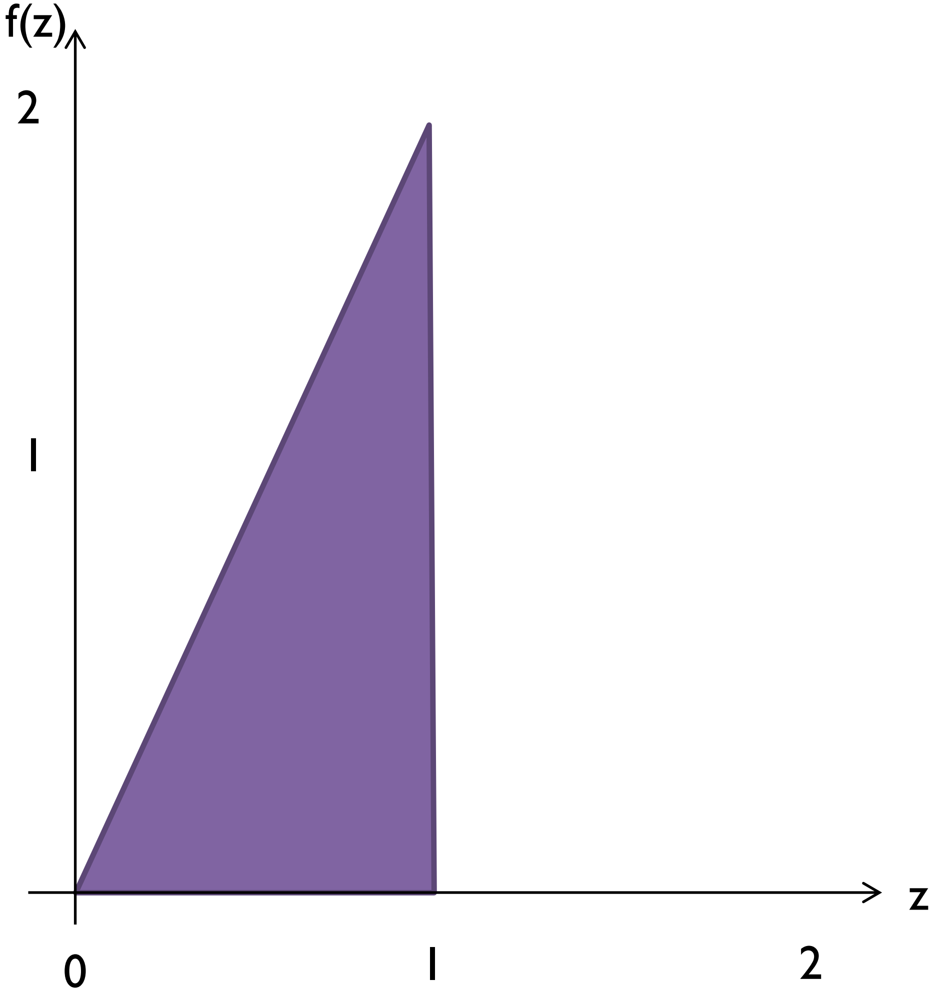

$$\begin{cases} 2Z & 0 \le Z < 1 \end{cases}$$

```
X := Uniform(0,1)
Y := Uniform(0,1)

Z := X + Y
observe Z < 1
return Z
```

$ psi sum_uniform.prb

Probabilistic Programs

Extend Standard (Deterministic) Programs

Distribution	`X := Uniform(0, 1);`
Assertion	`assert ( X >= 0 );`
Observation	`observe ( X >= 0.5 );`
Query	`return X;`

Probabilistic Programs

Extend Standard (Deterministic) Programs

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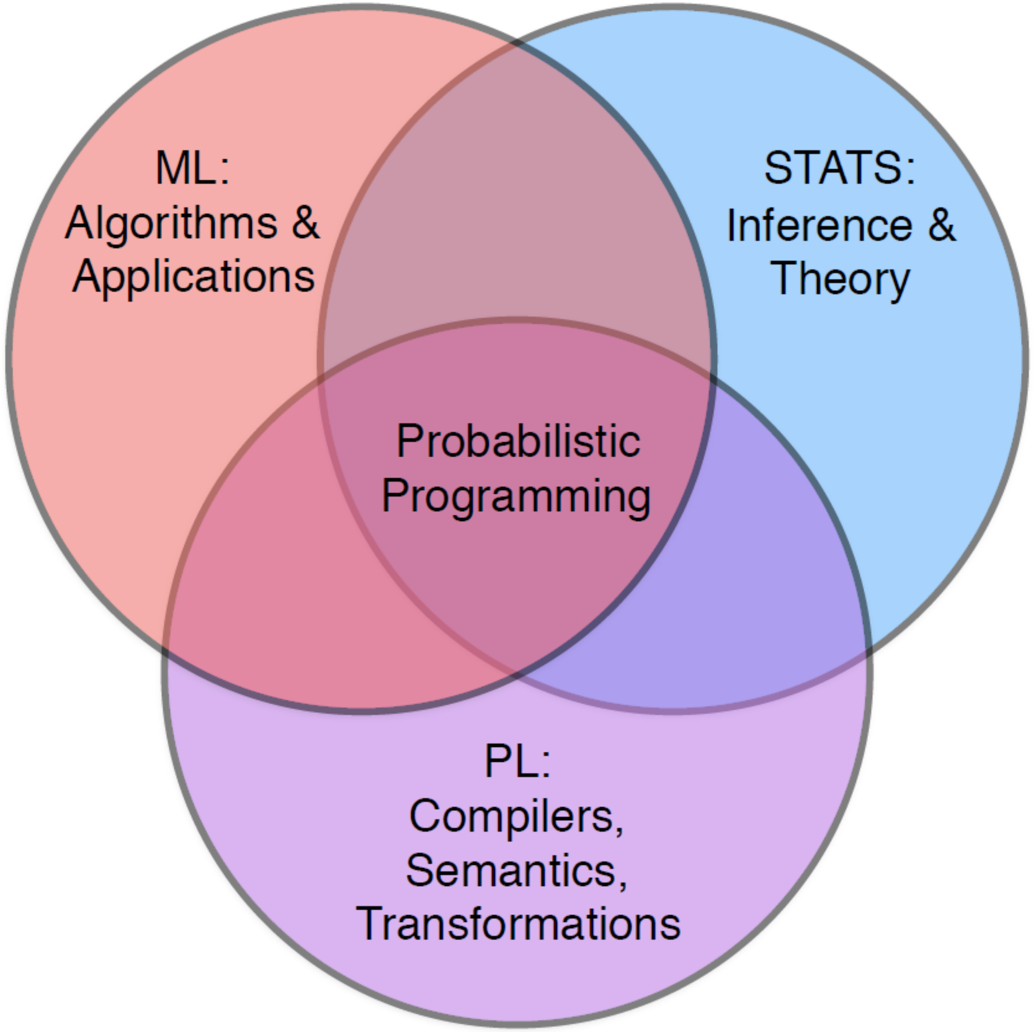
Probabilistic Programming

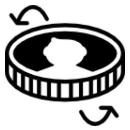


Graphic from Frank Wood's lecture

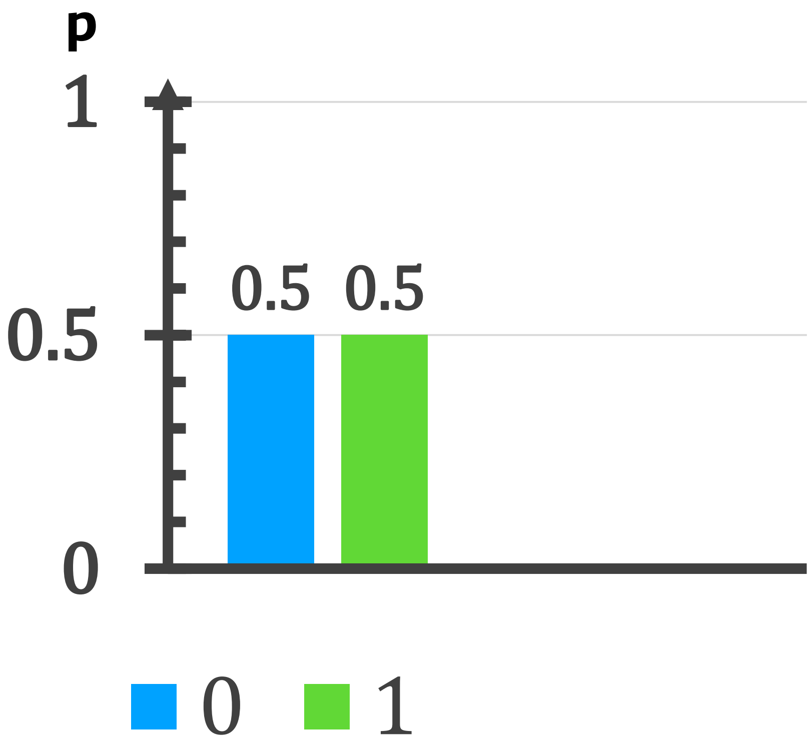
Probabilistic Model

$A \sim Bernoulli(0.5)$

$P(A = 1)$



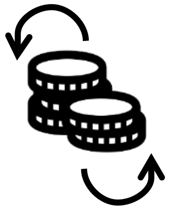
$head: 1$
$tail: \quad 0$



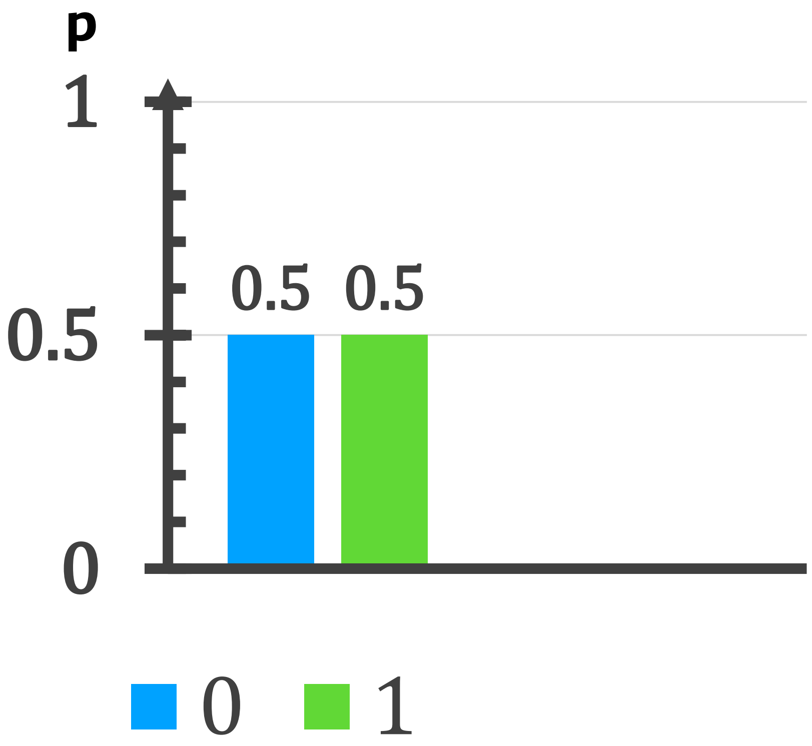
Probabilistic Model

$A \sim Bernoulli(0.5)$
$B \sim Bernoulli(0.5)$
$C \sim Bernoulli(0.5)$

$P(A = 1)$



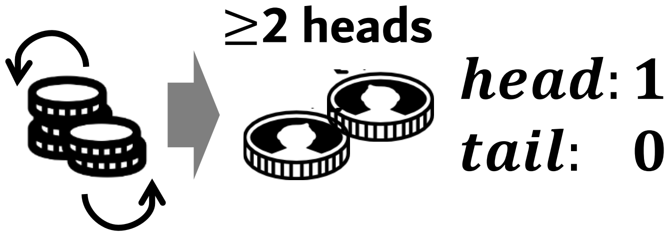
$head: 1$
$tail: \quad 0$

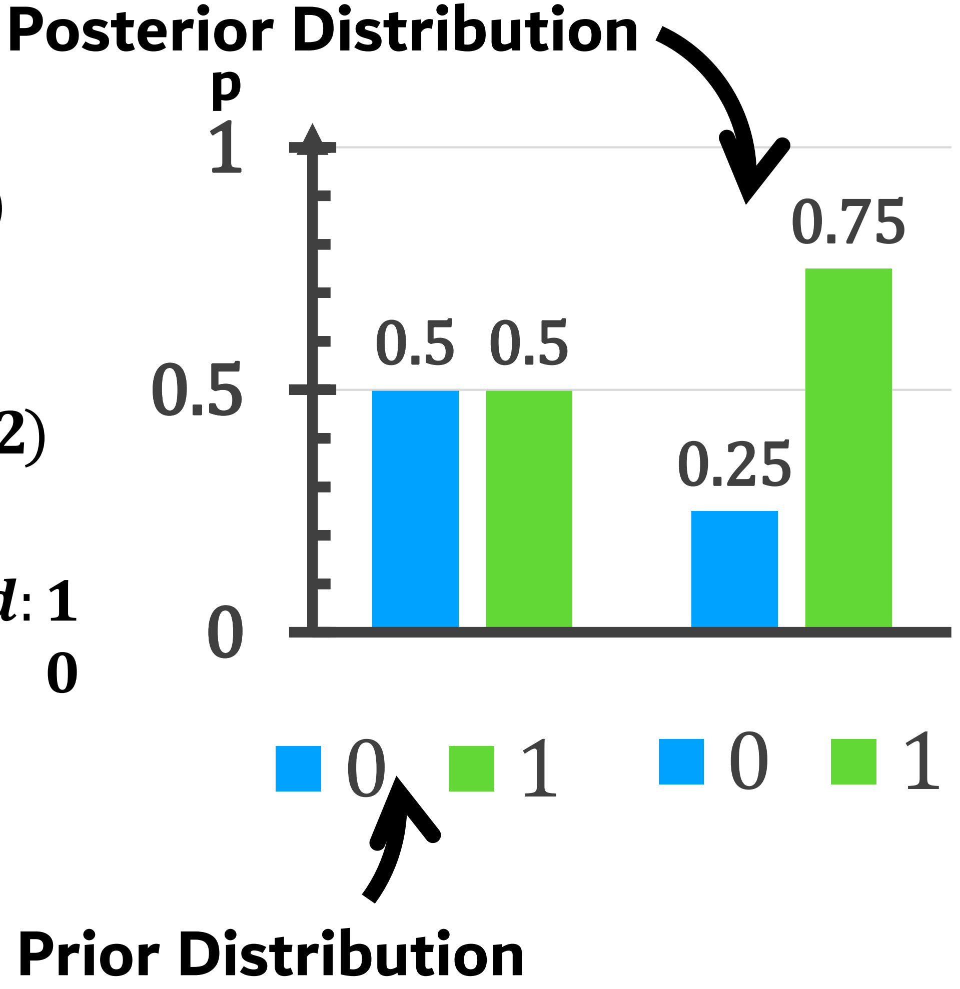


Probabilistic Model

$A \sim Bernoulli(0.5)$
$B \sim Bernoulli(0.5)$
$C \sim Bernoulli(0.5)$

$P(A = 1 | A + B + C \geq 2)$





Probabilistic Programming

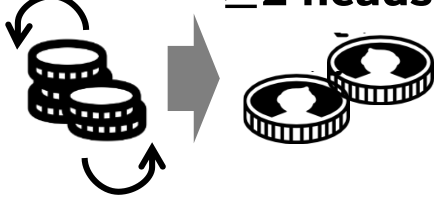
$A \sim Bernoulli(0.5)$
$B \sim Bernoulli(0.5)$
$C \sim Bernoulli(0.5)$

```
def main(){
    A:=flip(0.5);
    B:=flip(0.5);
    C:=flip(0.5);

    observe(A+B+C>=2);
    return A;
}
```

$P(A = 1 | A + B + C \geq 2)$

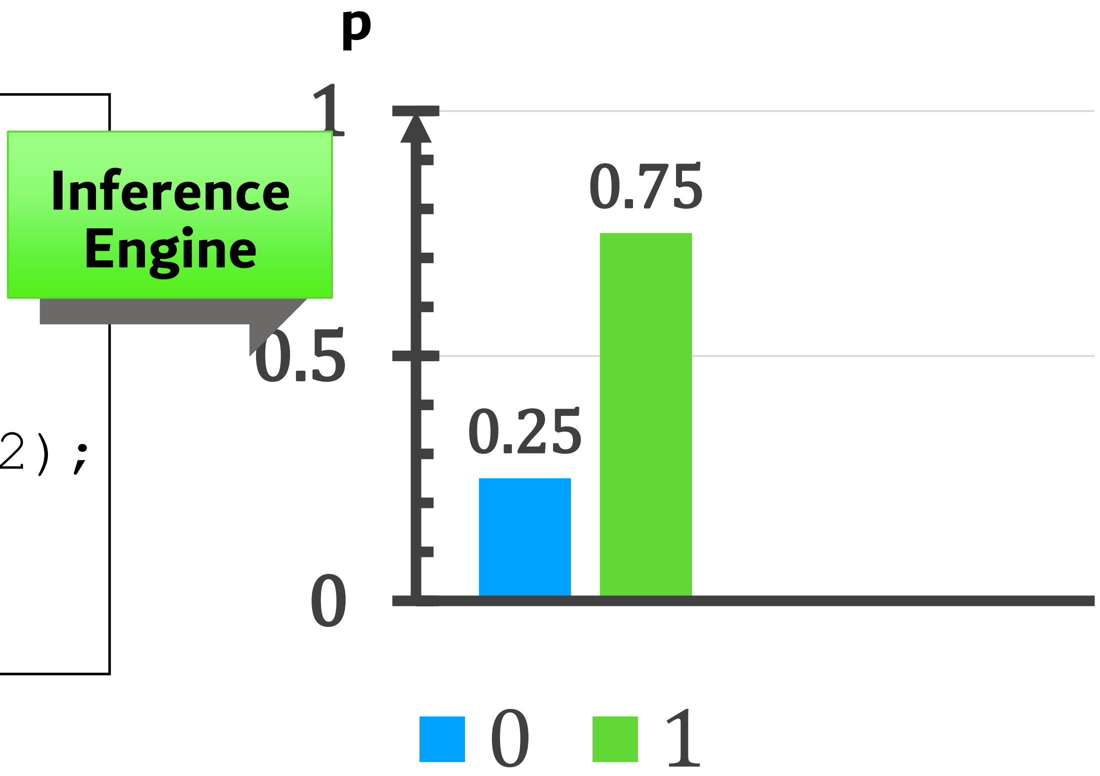
≥2 heads



$head$: 1
$tail$: 0

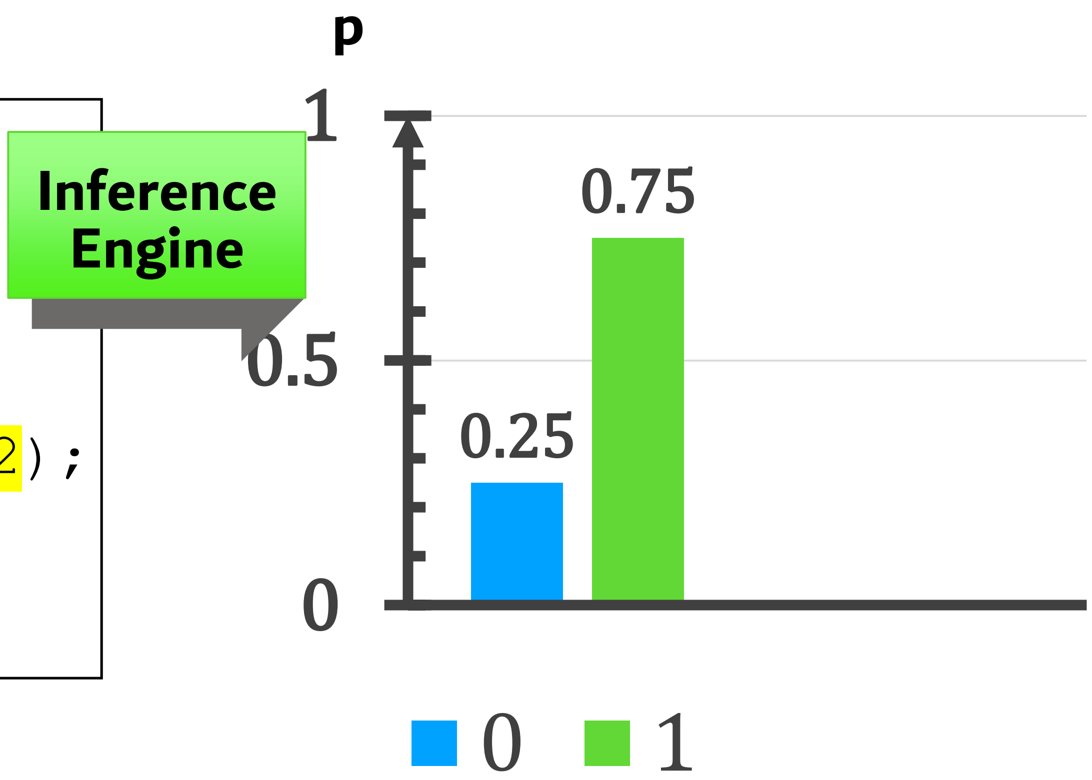
Probabilistic Programming

```
def main(){
   A:=flip(0.5);
   B:=flip(0.5);
   C:=flip(0.5);
   observe(A+B+C>=2);
   return A;
}
```



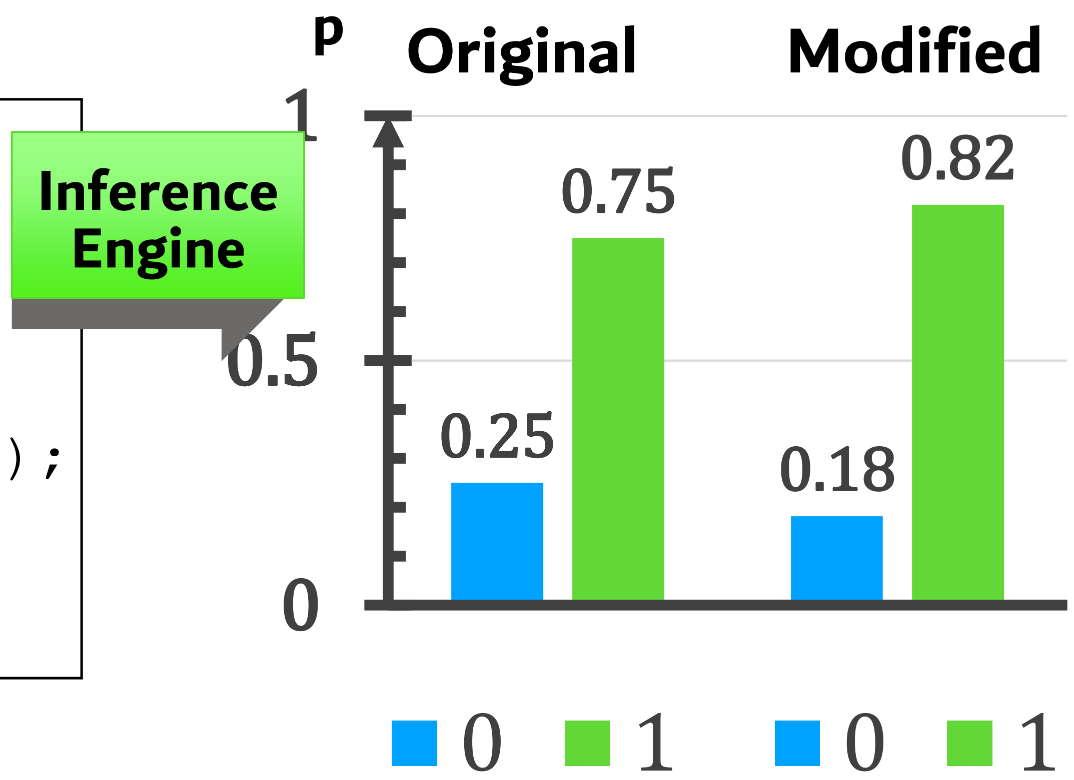
Probabilistic Programming

```
def main(){
    A:=flip(0.5);
    B:=flip(0.5);
    C:=flip(0.5);
    observe(A+B+C>=2);
    return A;
}
```



Probabilistic Programming

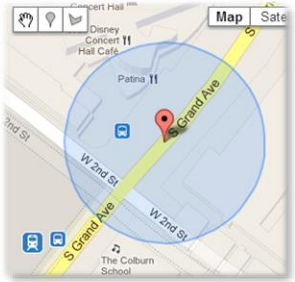
```
def main(){
    A:=flip(0.5+0.1);
    B:=flip(0.5);
    C:=flip(0.5);
    observe(A+B+C>=2);
    return A;
}
```



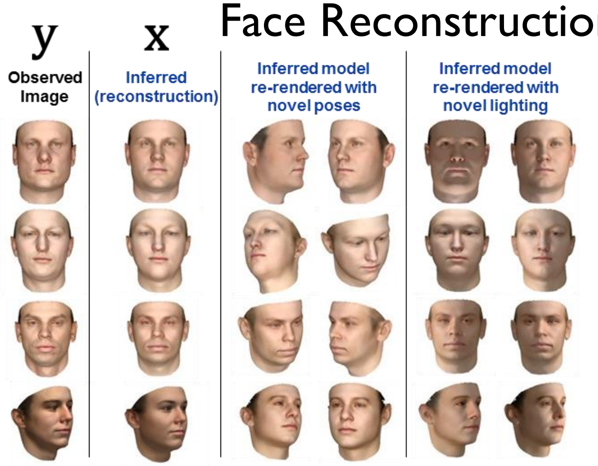
Probabilistic Applications

Modeling of
Complex Systems

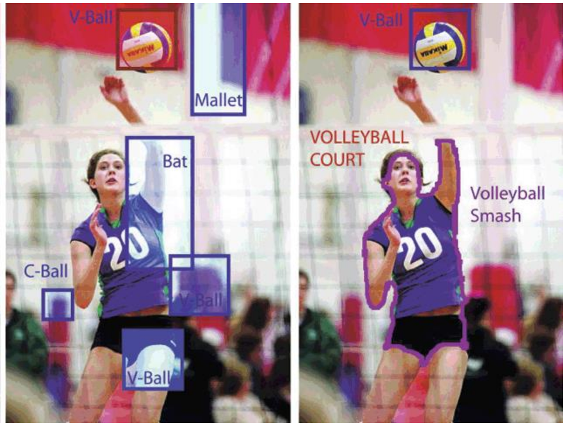
GPS & Navigation



Face Reconstruction



Scene labeling



Spam
Filter

Microsoft

XBOX

facebook

UBER

Pyro

Google

Tensorflow

Example Language:

WWW.WEBPPL.ORG

Probability Refresher

2.1. Basic definition.

We define a *probability triple* or *(probability) measure space* or *probability space* to be a triple $(\Omega, \mathcal{F}, \mathbf{P})$, where:

- the *sample space* Ω is any non-empty set (e.g. $\Omega = [0,1]$ for the uniform distribution considered above);
- the *σ-algebra* (read "sigma-algebra") or *σ-field* (read "sigma-field") $\mathcal{F}$ is a collection of subsets of Ω, containing Ω itself and the empty set $\emptyset$, and closed under the formation of complements* and countable unions and countable intersections (e.g. for the uniform distribution considered above, $\mathcal{F}$ would certainly contain all the intervals $[a, b]$, but would contain many more subsets besides);
- the *probability measure* $\mathbf{P}$ is a mapping from $\mathcal{F}$ to $[0,1]$, with $\mathbf{P}(\emptyset) = 0$ and $\mathbf{P}(\Omega) = 1$, such that $\mathbf{P}$ is countably additive as in (1.2.3).

Rosenthal J.; A First Look at Rigorous Probability Theory 2 ed.

Probability Refresher

Probability Distribution

- Discrete Distributions
- Continuous Distributions
- Hybrid Joint Distributions

Distribution Function

Probability Distribution Function

Probability Mass Function

Probability Density Function

Expectation

Expected value: measure of central tendency

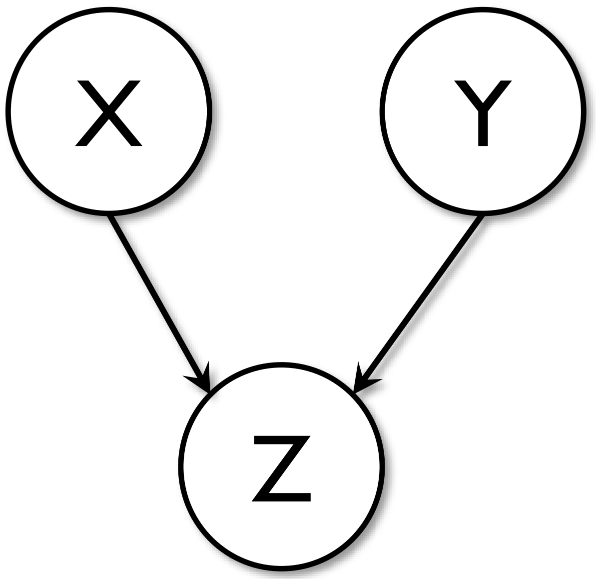
Variance: measure of spread

Probabilistic Programs and Graphical Models

```
X := Uniform(0,1)
Y := Uniform(0,1)

Z := X + Y

return Z
```



Dependency Graph

Bayes' Rule

Belief Revision

Thomas Bayes
1701 –1761

Bayes' Rule

Belief Revision

Hypothesis

$$\Pr(\theta \mid x) = \frac{\Pr(x \mid \theta) \cdot \Pr(\theta)}{\Pr(x)}$$

Data

Bayes' Rule

Belief Revision

Posterior Distribution

Likelihood

Prior Distribution

$$\Pr(\theta \mid x) = \frac{\Pr(x \mid \theta) \cdot \Pr(\theta)}{\Pr(x)}$$

Normalization Constant

Is Our Brain Statistical?*

Probability of sickness is 1%

If a patient is sick, the probability that medical test returns positive is 80% (true positive)

If a patient is not sick, the probability that medical test returns positive is 9.6% (false positive)

For a given patient, the test returned positive.

What is the probability that the patient is sick?

* Kahneman and Tversky (1974)

Is Our Brain Statistical?

```
var test_effective = function() {
 var PatientSick = flip(0.01);

 var PositiveTest =
   PatientSick? flip(0.8): flip(0.096);

 condition (PositiveTest == true);

 return PatientSick;
}

Infer ({method: 'enumerate'},
        test_effective)
```

Fallacy:
Base rate
neglect

For discussion: Goodman & Tenenbaum,
Probabilistic Models of Cognition (Ch. 3)

Bayesian Nets

Alternative representation of probabilistic models

Graphical representation of dependencies among random variables:

- Nodes are variables
- Links from parent to child nodes are direct dependencies between variables
- Instead of full joint distribution, now terms $\Pr(X | parents(X))$.

The graph has no cycles! DAG

Queries

Posterior distribution – what we got

Expected value – $$\mathbb{E}(\boldsymbol{X}) = \sum_{\boldsymbol{x} \in \boldsymbol{Dom}(\boldsymbol{X})} \boldsymbol{x} \cdot \Pr(\boldsymbol{x})$$

Most likely value – Mode of the distribution

Variable Dependencies

```
var test_effective = function() {
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 return PatientSick;
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Infer ({method: 'enumerate'},
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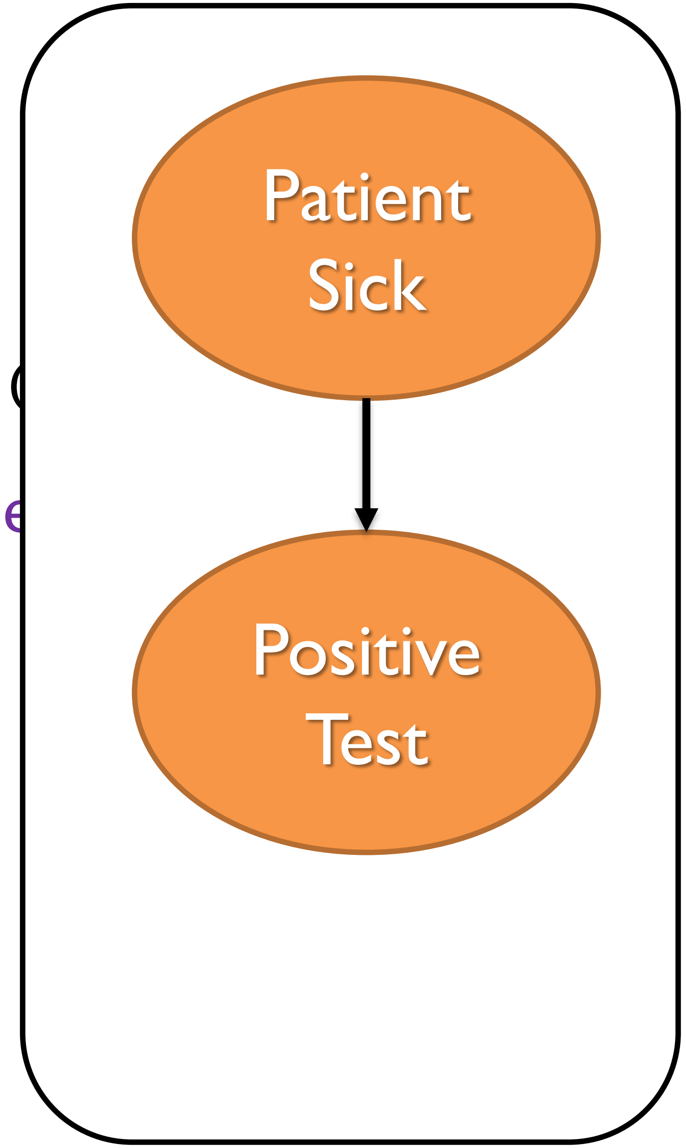


Variable Dependencies

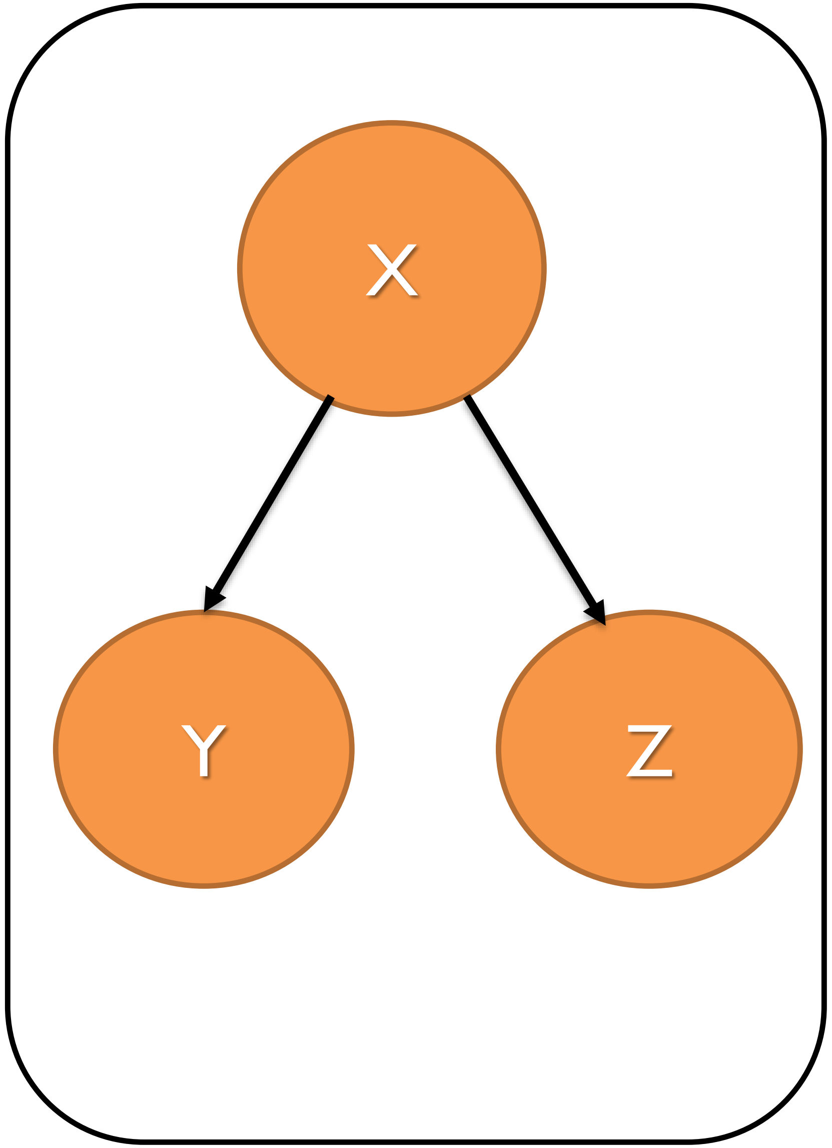
```
var test_x = function() {
 var x = flip(0.50);

 var y = x?
   flip(0.1): flip(0.2);

 var z = x?
    flip(0.3): flip(0.4);

 condition(x == 1)

 return [y, z]
}
```



Variable Dependencies

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 var z = x?
    flip(0.3): flip(0.4);


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```
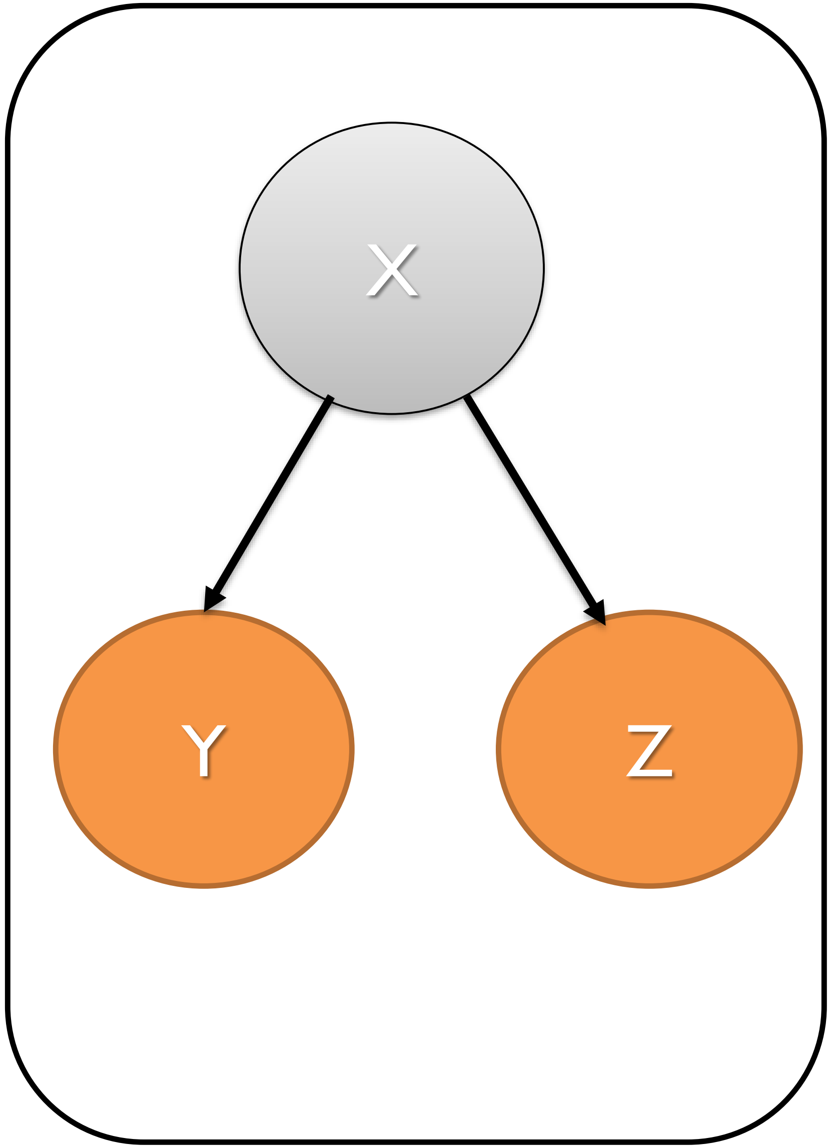



Reminder: Independence

Definition:

$$\boldsymbol{Pr(X, Y) = Pr(X) \cdot Pr(Y)}$$

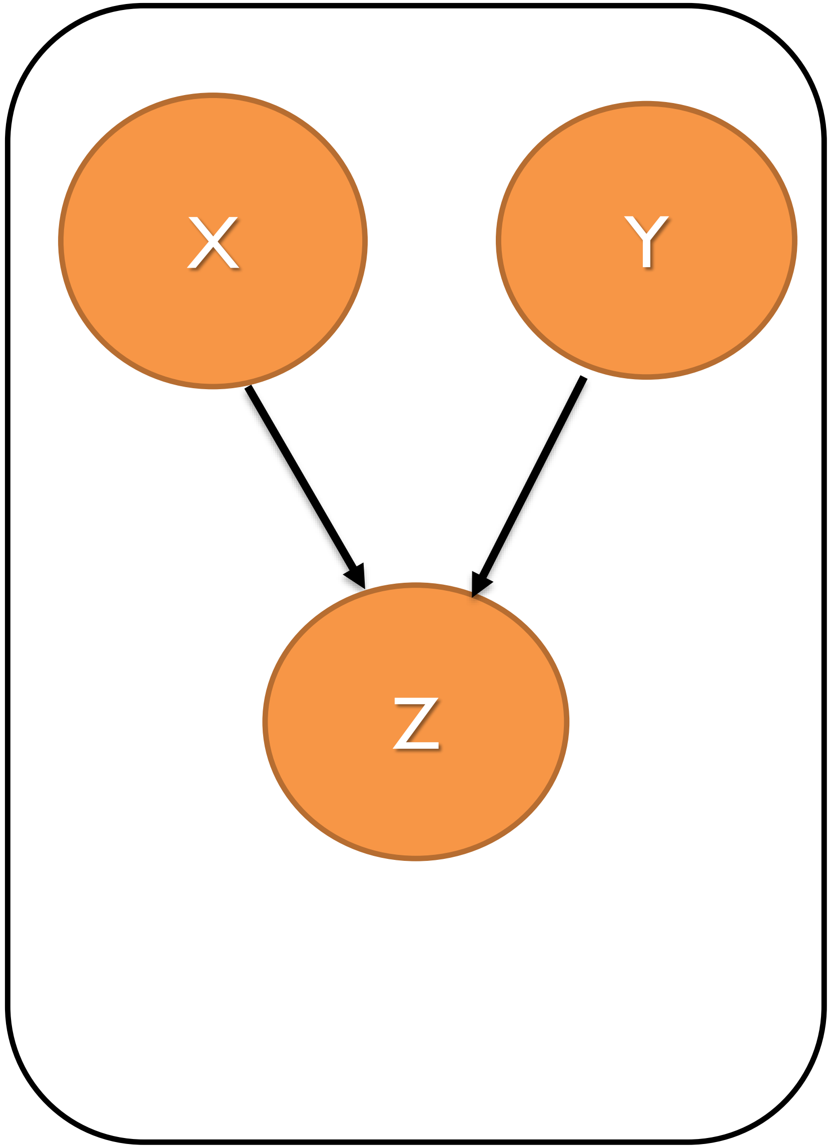
But also*:

$$\boldsymbol{Pr(X \mid Y) = Pr(X)}$$
$$\boldsymbol{Pr(Y \mid X) = Pr(Y)}$$

*Using the fact that for any two variables $\boldsymbol{Pr(X, Y) = Pr(X|Y) \cdot Pr(Y)}$

Variable Dependencies

```
var test_z = function(){
 var x = flip(0.50);

 var y = flip(0.1);

 var z = x+y;

 condition(z == 1);

 return x;
}
```



Variable Dependencies

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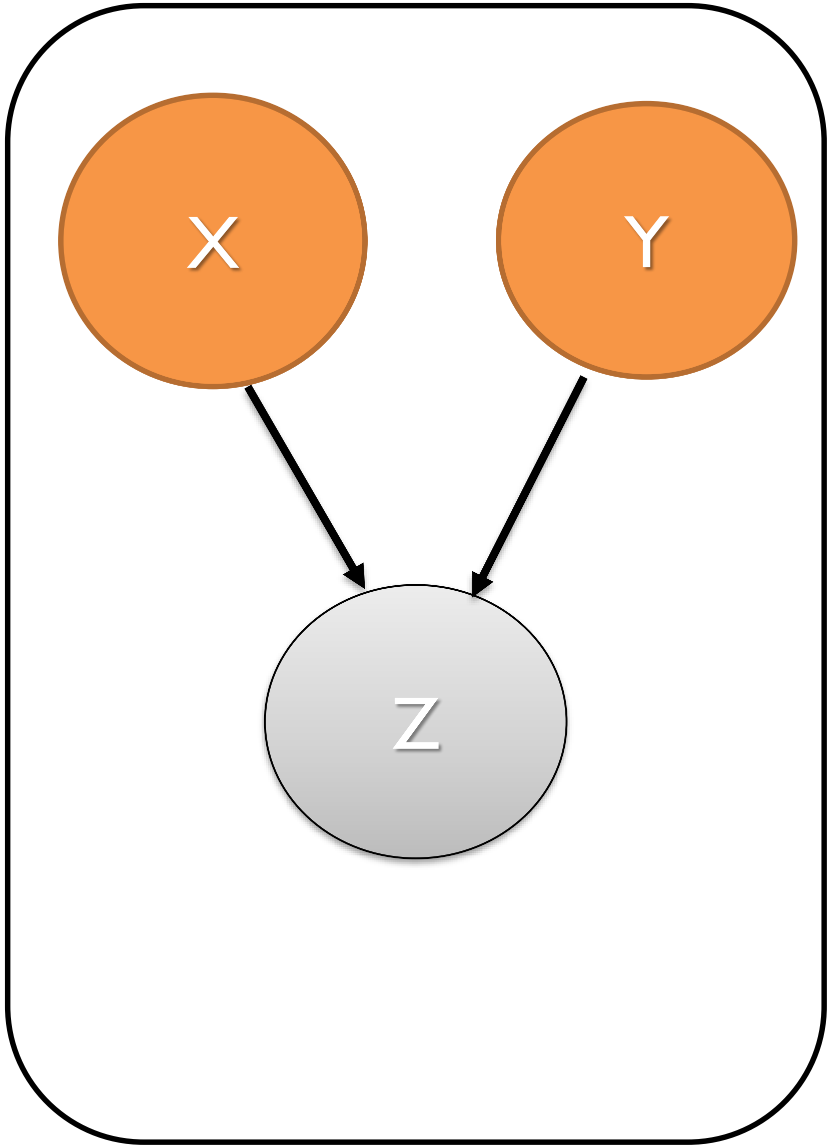

Bayes' Rule

Belief Revision

Likelihood

Prior Distribution

Posterior Distribution

$$\Pr(\theta \mid x) = \frac{\Pr(x \mid \theta) \cdot \Pr(\theta)}{\Pr(x)}$$

Normalization Constant

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$$\Pr(\theta \mid x) \sim \Pr(x \mid \theta) \cdot \Pr(\theta)$$

Enough to order different interpretations and select the most likely one

Bayes' Rule

Belief Revision

Posterior Distribution

Likelihood

Prior Distribution
Equvi-probable

$$\Pr(\theta \mid x) \sim \Pr(x \mid \theta) \cdot \Pr(\theta)$$

Enough to order different interpretations and select the most likely one

Bayes' Rule

Belief Revision

Likelihood

Posterior Distribution

$$\Pr(\theta \mid x) \sim \Pr(x \mid \theta)$$

Enough to order different interpretations and select the most likely one

Beyond Bayesian Net Models

Geometric Distribution: Probability of the number of Bernoulli trials to get one success

```
var geometric = function() {
  return flip(.5) ? 0 : geometric() + 1;
}

var dist = Infer({method: 'enumerate', maxExecutions: 10},
                 geometric);
viz.auto(dist);
```

Exact Inference

Naïve approach: Compute $P(x_1, x_2, \ldots, x_n)$

Better approach:

Take advantage of (conditional) independencies

- Whenever we can expose conditional independence, e.g., $\mathrm{P}(x_1, \mathrm{x}_2 | \mathrm{x}_3) = \mathrm{P}(\mathrm{x}_1 | \mathrm{x}_3) \cdot \mathrm{P}(\mathrm{x}_2 | \mathrm{x}_3)$ the computation is more efficient

Compute distributions from parents to children

Complexity of Exact Inference

Number of variables: $\boldsymbol{n}$

Naïve enumeration: complexity is $O(2^n)$

Variable Elimination: if the maximum number of parents of the nodes is $k \in \{1, \dots, n\}$, then the complexity is $n \cdot O(2^k)$.

For many models this is a good improvement, but always possible to construct pathological models.

Continuous Models

TrueSkill:

- Measure player skills in various sports

Each player has an unknown parameter skill that cannot be directly measured (i.e., it is hidden)

What we can observe is how the in-game performance of the player (which depends on the skill) compares to the performance of the other player

TrueSkill Model

Player skill: initially, we assume all players have similar (randomly assigned) skills, centered around some average:

$$Skill \sim Gaussian(100,10)$$

Player performance: it is based on the skill, but can be either higher or lower, depending on the moment of inspiration:

$$Perf \sim Gaussian(Skill, 15)$$

Tournament scores: Each player plays against each other, we can observe that a player with better performance won

$$PerfPlayerA > PerfPlayerB$$

TrueSkill Example

```
var trueSkill = function(){

  var skillA = gaussian(100, 10);
  var skillB = gaussian(100, 10);
  var skillC = gaussian(100, 10);

  var perfA1 = gaussian(skillA, 15), perfB1 = gaussian(skillB, 15);
  condition (perfA1 > perfB1);

  var perfB2 = gaussian(skillB, 15), perfC2 = gaussian(skillC, 15);
  condition (perfB2 > perfC2);

  var perfA3 = gaussian(skillA, 15), perfC3 = gaussian(skillC, 15);
  condition (perfA3 > perfC3);

  return skillA;
}
```

```
var res = Infer({method: 'MCMC', samples:
                      50000}, trueSkill)
print("Expected value: "+expectation(res));
viz.auto(res);
```

Inference with Continuous and Hybrid Models *(Exact and Approximate)*

Sampling (Rejection – Church)

Sampling (MCMC – Church & Stan & Figaro)

Variational Inference (Fun & Infer.NET)

Exact Symbolic (PSI & Hakaru)