Probabilistic & Approximate Computing

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Previously:

We looked into various ways to use probabilistic programs!

Coming up next:
Let’s define our simple probabilistic language!

Operational Semantics *(deterministic)*

Simulates how the program executes on an abstract state-machine

X	Y
3	0

PC
32

```
31:   ...
32:   y = x + 2
33:   ...
```

X	Y
3	5

PC
33

Two flavors:

- Small-step operational semantics

  read from location pointed to by x, do addition, store to location pointed to by y

- Big-step operational semantics

  x + 2 evaluates to the value 5; store this value to the location pointed to by y

Operational Semantics *(deterministic)*

Execution Stages:

- Map the program and inputs to the ***initial configuration***

```
init:    (x, 3)
01:  def program(x){…}
```

X	Y
3	0

PC
01

- ***Execute the steps*** that represent the instructions of the abstract machine

```
32:  y = x + 2
```

X	Y
3	5

PC
33

- Map the ***final configuration*** (if it exists) to the output

```
33:  return y
```

X	Y
3	5

PC
33

Simple Deterministic Language

$$
\begin{array}{lcl}
x & \in & \text{Vars} \\
\mathcal{T} & ::= & \texttt{bool} \\
\textbf{uop} & ::= & \texttt{not} \\
\textbf{bop} & ::= & \texttt{and} \mid \texttt{or} \\
\\
\mathcal{D} & ::= & \mid \mathcal{T}\, x_1, x_2, \ldots, x_n \\
\\
\mathcal{E} & ::= & \\
 & & x \\
 & & \mid c \\
 & & \mid \mathcal{E}_1 \;\textbf{bop}\; \mathcal{E}_2 \\
 & & \mid \textbf{uop}\; \mathcal{E}_1
\end{array}
$$

$$
\begin{array}{lcl}
\mathcal{S} & ::= & \\
 & & x := \mathcal{E} \\
 & & \mid \mathcal{S}_1 ; \mathcal{S}_2 \\
 & & \mid \texttt{if}\ \mathcal{E}\ \texttt{then}\ \mathcal{S}_1\ \texttt{else}\ \mathcal{S}_2 \\
 & & \mid \texttt{while}\ \mathcal{E}\ \texttt{do}\ \mathcal{S}_1 \\
\\
\mathcal{P} & ::= & \mathcal{D}\ \mathcal{S}
\end{array}
$$

Subset of Language from Claret et al. Bayesian inference using data flow analysis (FSE'13)

Operational Semantics

Tuple: $S = (\mathcal{C}, \rightarrow, \mathcal{C}_{final}, \mathcal{I}, \mathcal{O})$

- $\mathcal{C}$: Set of configurations (e.g.,$\mathcal{C} = Stmt \times Stack \times Mem$)
- $\rightarrow$: Transition relation, which defines possible transitions between the configurations
  - Deterministic if for each start configuration $c \in \mathcal{C}$ there exists a single result configuration $c' \in \mathcal{C}$
  - Nondeterministic if there can be multiple $c \in \mathcal{C}$
- $\mathcal{C}_{final}$: Set of final configurations ($\mathcal{C}_{final} \subseteq \mathcal{C}$) in which the program successfully ends
- $\mathcal{I}$ maps program source and input to initial configuration $\mathcal{C}_0$
- $\mathcal{O}$ maps final configuration $\mathcal{C}_{final}$ to the output

Operational Semantics *(small step)*

Configuration: $c \in \mathcal{C} ::= Stmt \times \Sigma$

$\sigma \in \Sigma ::= (x_1 \to v_1, \dots x_n \to v_n)$

$Stmt$ – next statement or no statement to run $(\cdot)$

x_i – variables, v_i – values

Final Configuration:

- Has the form `(skip, σ) = (skip,` $(x_1 \to v_1, \dots x_n \to v_n))$
- Specifies configurations in which program terminated normally
- But, execution can get stuck (there is a statement for which there is no transition)
- Or, execution may never terminate (the execution loops infinitely)

Operational Semantics *(small step)*

For expressions: $\rightarrow_b \in Expr \times \Sigma \rightarrow Expr$

e1, e2 – expressions
v1, v2 – values

$(\mathtt{x},\ \sigma) \rightarrow_b \sigma(x)$

Reading a variable

$(\mathtt{uop\ v1},\ \sigma) \rightarrow_b \mathtt{v2}$

Unary operation on constants
(v2 is the result of unary operation)

$$\frac{(\mathtt{e1},\ \sigma) \rightarrow_b \mathtt{e1'}}{(\mathtt{uop\ e1},\ \sigma) \rightarrow_b \mathtt{uop\ e1'}}$$

Unary operation on subexpressions

$(\mathtt{v1\ bop\ v2},\ \sigma) \rightarrow_b \mathtt{v3}$

Binary operation on constants
(v3 is the result of binary operation)

$$\frac{(\mathtt{e1},\ \sigma) \rightarrow_b \mathtt{e1'}}{(\mathtt{e1\ bop\ e2},\ \sigma) \rightarrow_b \mathtt{e1'\ bop\ e2}}$$

Binary operation on subexpressions

$$\frac{(\mathtt{e2},\ \sigma) \rightarrow_b \mathtt{e2'}}{(\mathtt{v1\ bop\ e2},\ \sigma) \rightarrow_b \mathtt{v1\ bop\ e2'}}$$

Binary operation on subexpressions

Operational Semantics *(small step)*

For statements: $\rightarrow \in \mathcal{C} \rightarrow \mathcal{C}$

e1, e2 – expressions
v1, v2 – values

$(\texttt{x = v, } \sigma) \rightarrow (\texttt{skip}, \sigma[x \leftarrow v])$ — *Assigning a constant*

$(\texttt{x = e, } \sigma) \rightarrow (\texttt{x = e}', \sigma)$
where $(\texttt{e, } \sigma) \rightarrow_b \texttt{ e'}$ — *Assigning an expression*

$(\texttt{skip; s1, } \sigma) \rightarrow (\texttt{s1}, \sigma)$ — *Sequence rule (1)*

$(\texttt{s1; s2, } \sigma) \rightarrow (\texttt{s1'; s2}, \sigma')$
where $(\texttt{s1, } \sigma) \rightarrow (\texttt{s1', } \sigma')$ — *Sequence rule (2)*

Operational Semantics *(small step)*

For statements: $\rightarrow \;\in \mathcal{C} \rightarrow \mathcal{C}$

e1, e2 – expressions
v1, v2 – values

$(\texttt{if true then s1 else s2, } \sigma) \rightarrow (\texttt{s1, } \sigma)$ ***Conditional (then)***

$(\texttt{if false then s1 else s2, } \sigma) \rightarrow (\texttt{s2, } \sigma)$ ***Conditional (else)***

$$\frac{(\texttt{e1, } \sigma) \rightarrow_b \texttt{e1'}}{(\texttt{if e then s1 else s2, } \sigma) \rightarrow (\texttt{if e' then s1 else s2, } \sigma)}$$ ***Conditional (expr)***

While loop

$(\texttt{while e do s, } \sigma) \rightarrow$
$\quad(\texttt{if e then \{s1; while e do s\} else skip, } \sigma)$

Simple Probabilistic Language

$$
\begin{array}{lcl}
r & \in & \mathbb{R} \\
x & \in & \text{Vars} \\
\mathcal{T} & ::= & \texttt{bool} \\
\textbf{uop} & ::= & \texttt{not} \\
\textbf{bop} & ::= & \texttt{and} \mid \texttt{or} \\
\\
\mathcal{D} & ::= & \mid \mathcal{T} x_1, x_2, \ldots, x_n \\
\\
\mathcal{E} & ::= & \\
 & & x \\
 & & \mid c \\
 & & \mid \mathcal{E}_1 \textbf{ bop } \mathcal{E}_2 \\
 & & \mid \textbf{uop } \mathcal{E}_1
\end{array}
$$

$$
\begin{array}{lcl}
\mathcal{S} & ::= & \\
 & & x := \mathcal{E} \\
 & & \mid x := \texttt{Bernoulli}(r) \\
 & & \mid \texttt{observe } (\mathcal{E}) \\
 & & \mid \texttt{skip} \\
\\
 & & \mid \mathcal{S}_1 ; \mathcal{S}_2 \\
 & & \mid \texttt{if } \mathcal{E} \texttt{ then } \mathcal{S}_1 \texttt{ else } \mathcal{S}_2 \\
 & & \mid \texttt{while } \mathcal{E} \texttt{ do } \mathcal{S}_1 \\
\\
\mathcal{P} & ::= & \mathcal{D}\ \mathcal{S}
\end{array}
$$

Claret et al. Bayesian inference using data flow analysis (FSE'13)

Probabilistic State

Deterministic

State: $\sigma_0 \in \Sigma_0 ::= (x_1 \to v_1, \ldots x_n \to v_n)$

x_i –variables $\quad v_i$ – values

Probabilistic

State: $\sigma \in \Sigma = \Sigma_0$

Expressions: $A \in \mathcal{E} \times \Sigma \to \text{Bool}$

Statements: $T \subseteq (S \times \Sigma) \times (\Sigma \times [0,1])$

(we use notation $(\cdot,\cdot) \xrightarrow{\cdot} \cdot$)

Probabilistic Assignment

Statements: $T \subseteq (S \times \Sigma) \times (\Sigma \times [0,1])$

$(\texttt{x = v, } \sigma) \rightarrow (\texttt{skip}, \sigma[x \leftarrow v])$ ***Assigning a constant***

$(\texttt{x = e, } \sigma) \rightarrow (\texttt{x = e}', \sigma)$ ***Assigning an expression***

where $(\texttt{e, } \sigma) \rightarrow_b \texttt{ e'}$

$$(x = \texttt{Bern}(p_B), \sigma) \xrightarrow{p_B} (\texttt{skip}, \sigma[x \leftarrow \text{True}])$$
$$(x = \texttt{Bern}(p_B), \sigma) \xrightarrow{1-p_B} (\texttt{skip}, \sigma[x \leftarrow \text{False}])$$

Example: Bernoulli Program

(

X	Y
--	--

, 1.0)

```
X := Bernoulli(0.7);
```

(

X	Y
True	--

, 0.7), (

X	Y
False	--

, 0.3)

```
Y := not X;
```

(

X	Y
True	False

, 0.7), (

X	Y
False	True

, 0.3)

Probabilistic Control Flow

$$(\texttt{skip; s1, } \sigma) \xrightarrow{1} (\texttt{s1}, \sigma)$$

Sequence rule (1)

$$\frac{(\texttt{s1, } \sigma) \xrightarrow{p_1} (\texttt{s1', } \sigma')}{(\texttt{s1; s2, } \sigma) \xrightarrow{p_1} (\texttt{s1'; s2, } \sigma')}$$

Sequence rule (2)

Recall:

$$(x = \texttt{Bern}(p_B), \sigma) \xrightarrow{p_B} (\texttt{skip}, \sigma[x \leftarrow \text{True}])$$

$$(x = \texttt{Bern}(p_B), \sigma) \xrightarrow{1-p_B} (\texttt{skip}, \sigma[x \leftarrow \text{False}])$$

Example: Bernoulli Program

(

X	Y
--	--

, 1.0)

```
X := Bernoulli(0.7);
```

(

X	Y
True	--

, 0.7), (

X	Y
False	--

, 0.3)

```
Y := Bernoulli(0.7);
```

(

X	Y
True	True

, 0.49), (

X	Y
True	False

, 0.21), (

X	Y
False	True

, 0.21), (

X	Y
False	False

, 0.09)

Probability of a Trace

(Finite) trace of executing a statement/program ($c \in \mathcal{C}$):

$$\theta = c_1 \xrightarrow{p_1} c_2 \xrightarrow{p_2} c_3 \xrightarrow{p_3} \dots \xrightarrow{p_n} c_{n+1}$$

- $c_1 = (S, \sigma_{init})$ is an initial configuration
- $c_{n+1} = (\texttt{skip}, \sigma_{final})$ is the final configuration, assuming that S terminates
- The execution took specific transitions from c_1 to c_{n+1}

Probability of the trace: $\Pr(\theta) = p_1 \cdot p_2 \cdot \dots \cdot p_n$

Probability of a Trace

(Finite) trace of executing a statement/program S ($c \in \mathcal{C}$):

$$\theta = c_1 \xrightarrow{p_1} c_2 \xrightarrow{p_2} c_3 \xrightarrow{p_3} \ldots \xrightarrow{p_n} c_{n+1}$$

Probability we end up in a specific configuration:

$$\Pr(c_{start}, c_{end}) = \sum_{\theta \in \mathcal{T}(c_{start}, c_{end})} \Pr(\theta)$$

where $\mathcal{T}(c_{start}, c_{end})$ is a set of all traces that start in c_{start} and finish in c_{end}

Aggregate trace semantics:

$$(S, \sigma_0) \overset{p}{\Longrightarrow} (S', \sigma') \quad \text{where } \mathrm{p} = \Pr\big((S, \sigma_0), (S', \sigma')\big)$$

Hint: always check whether the distribution on traces is discrete

Example: Bernoulli Program

(

X	Y
--	--

, 1.0)

```
X := Bernoulli(0.7);
```

(

X	Y
True	--

, 0.7), (

X	Y
False	--

, 0.3)

```
Y := Bernoulli(0.7);
```

(

X	Y
True	True

, 0.49), (

X	Y
True	False

, 0.21), (

X	Y
False	True

, 0.21), (

X	Y
False	False

, 0.09)

Observations

$$(\text{observe True}, \sigma) \xrightarrow{1} (\text{skip}, \sigma)$$

$$(\text{observe False}, \sigma) \xrightarrow{1} \emptyset$$

(doesn't exist: no such transition)

Normalization: If the probabilities $p_1 \ldots p_k$ do not sum up to 1, rescale them so that they do:

1. Compute sum: $Z = p_1 + p_2 + \cdots + p_k$
2. Rescale: $p_1' = \frac{p_1}{Z}, p_2' = \frac{p_1}{Z}, \ldots p_k' = \frac{p_k}{Z}$

Do only at the end of the program – although still expensive

Example: Bernoulli Program

(

X	Y
--	--

, 1.0)

```
X := Bernoulli(0.7);
Y := Bernoulli(0.7);
```

(

X	Y
True	True

, 0.49), (

X	Y
True	False

, 0.21), (

X	Y
False	True

, 0.21), (

X	Y
False	False

, 0.09)

```
observe ( X == True );
return Y;
```

(

X	Y
True	True

, 0.49/0.7), (

X	Y
True	False

, 0.21/0.7),

Example: Bernoulli Program

(

X	Y
--	--

, 1.0)

```
X := Bernoulli(0.7);
Y := Bernoulli(0.7);
```

(

X	Y
True	True

, 0.49), (

X	Y
True	False

, 0.21), (

X	Y
False	True

, 0.21), (

X	Y
False	False

, 0.09)

```
observe ( X == Y );
return Y;
```

(

X	Y
True	True

, 0.49/0.58), (

X	Y
False	False

, 0.09/0.58)

Denotational Semantics

Syntactic Domain: describes the syntax of the language (grammar): elements are e.g., nodes in the abstract syntax tree (AST)

Semantic Domain: mathematical entities and operations on them

- For instance, sets of numbers, sets of tuples
- For probabilistic programs expectations are easy to envision

Meaning Function: Translates elements of the syntactic domain to the elements and operations in the semantic domain

- Compositionality: the meaning of the AST is composite of the meaning of its nodes

Denotational Semantics for Straight-line Probabilistic Programs*

Meaning function: $[\![S]\!](f): (\Sigma \to R_{\geq 0}) \to (\Sigma \to R_{\geq 0})$

- S is the syntactic statement
- Function f: $\Sigma \to R_{\geq 0}$ is the expectation of a program state $\sigma \in \Sigma$
- The meaning function transforms the function f

$$[\![\text{Skip}]\!](f) = \lambda\sigma.\, f(\sigma)$$

$$[\![\text{x}{=}e]\!](f) = \lambda\sigma.\, f(\sigma[x \leftarrow v])$$ where e evaluates to v starting from σ

$$[\![\text{x}{=}\text{Bern}(p)]\!](f) = \lambda\sigma.\, (\, p \cdot f(\sigma[x \leftarrow 1]) + (1-p) \cdot f(\sigma[x \leftarrow 1])$$

* For full description, see: "Probabilistic Programming", Gordon, Henzinger, Nori, Rajamani, ICSE FoSE 2014

Example: Bernoulli Program

f: (

X	Y
--	--

, 1.0)

```
X := Bernoulli(0.7);
```

f: (

X	Y
True	--

, 0.7), (

X	Y
False	--

, 0.3)

```
Y := not X;
```

f: (

X	Y
True	False

, 0.7), (

X	Y
False	True

, 0.3)

Example: Bernoulli Program

f: (

X	Y
--	--

, 1.0)

```
X := Bernoulli(0.7);
```

f: (

X	Y
True	--

, 0.7), (

X	Y
False	--

, 0.3)

```
Y := Bernoulli(0.7);
```

f: (

X	Y
True	True

, 0.49), (

X	Y
True	False

, 0.21), (

X	Y
False	True

, 0.21), (

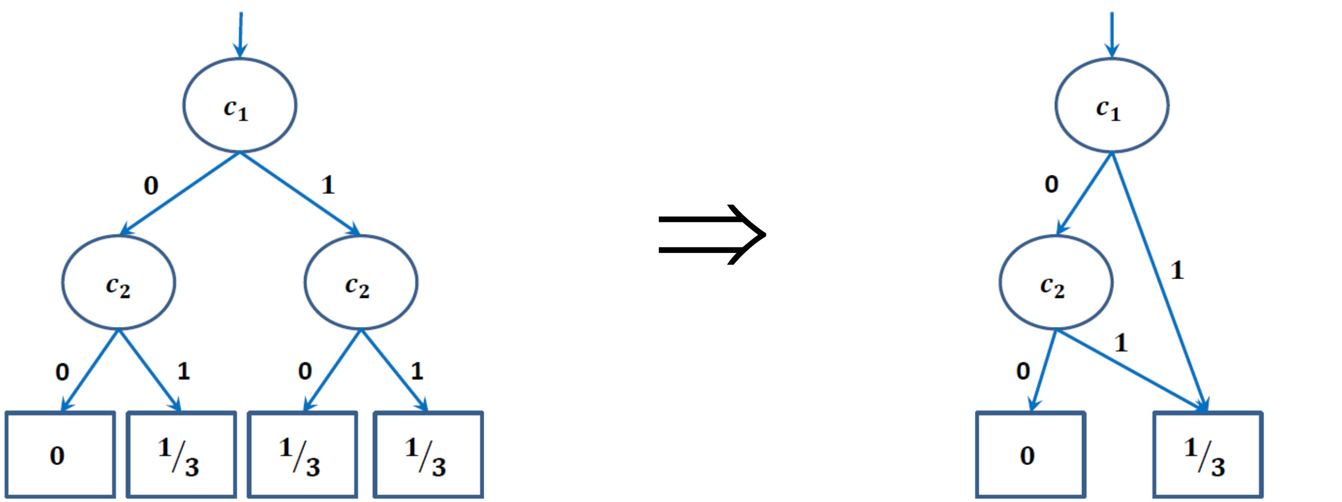
X	Y
False	False

, 0.09)

Implementation

Challenge: Many traces with many repeating subcomputations

Solution: Use Algebraic Decision Diagrams



Approximations

Challenge #1: Continuous Variables

- Discretize

Challenge #2: Large State (representation of ADD)

- Replace joint distribution with marginal

Challenge #3: Potentially infinite loops

- Heuristically stop the computation if the KL divergence is small ("In our implementation we terminate the fixpoint when the KL-divergence between c and p goes below a certain threshold.")

Evaluation

Question #1: Does the exact approach work?

Question #2: How much does it scale?

- How much does ADD representation help?

Question #3: How do approximations cover accuracy-scalability tradeoff space?

- How does it compare against the baseline results?